Recall: A <u>sequence</u> is an infinite list of numbers

$a_1, a_2, a_3, \ldots$

doesn't need to start w/ a_1

A <u>recurrence relation</u> is a formula for a_n in terms of (some of) $a_1, a_2, \ldots, a_{n-1}$.

Given a recurrence rel'n and <u>initial condition(s)</u>,

try to <u>solve</u>: explicit formula for a_n.

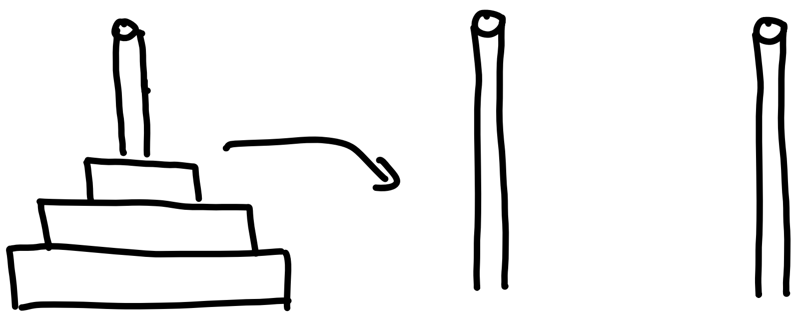
Ex 2: Towers of Hanoi:

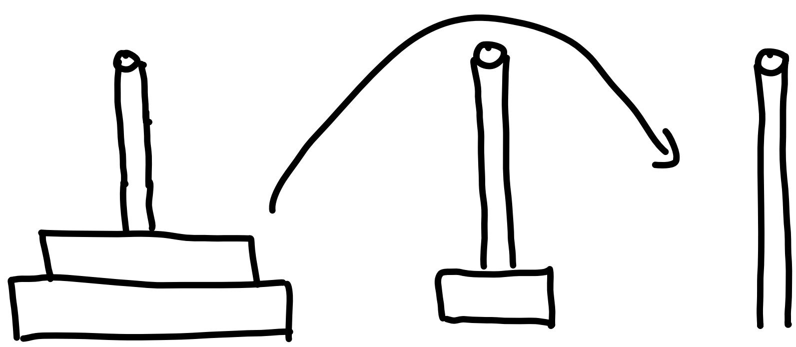
3 pegs

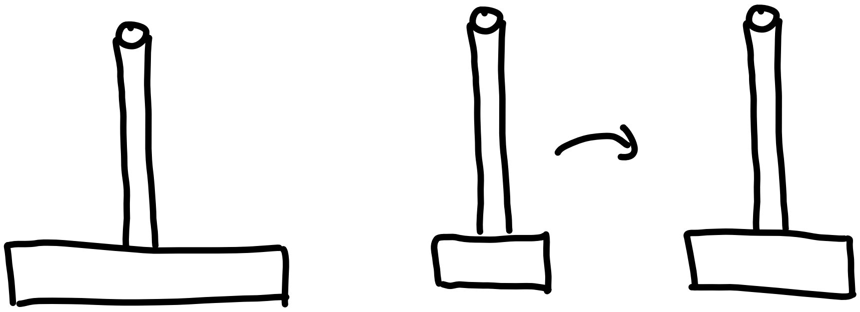
n discs of different sizes on Post 1

Want to move them all to Post 2

Can only stack smaller on larger





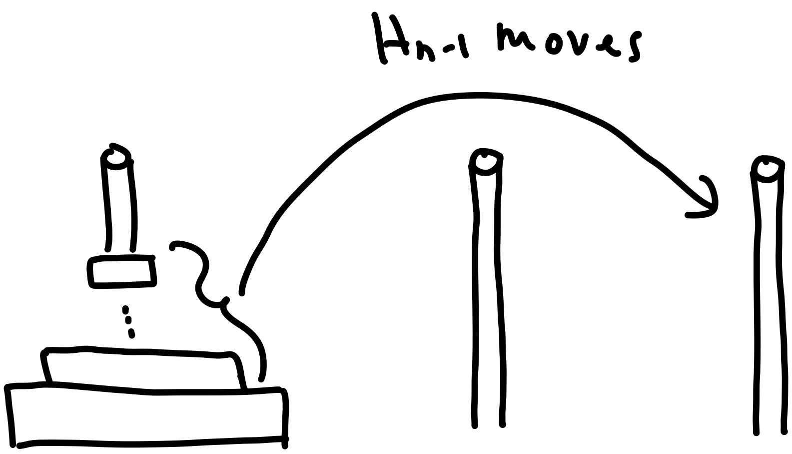


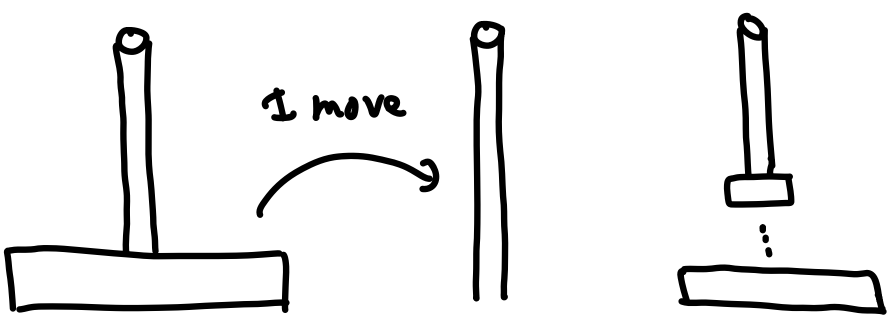
Class activity:

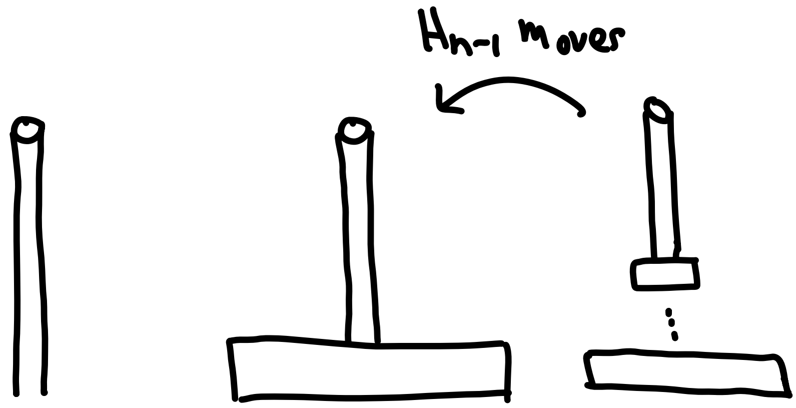
Find the minimum number of moves to move all 3 discs from Post 1 to Post 2

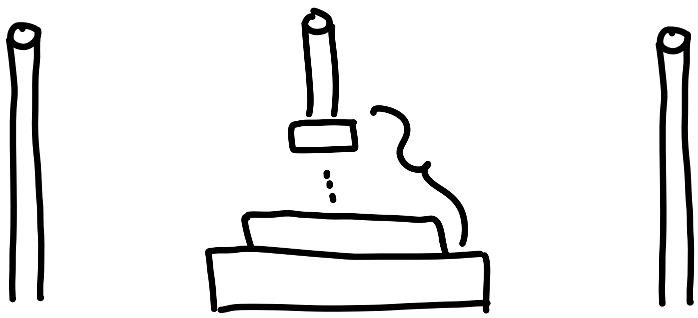
Let H_n be the min. num. moves to move all discs from Post 1 $\rightarrow$ Post 2

Now let's find a recurrence for H_n









Recurrence rel'n: $H_n = 2H_{n-1} + 1$

Initial cond.: $H_1 = 1$

n	H_n
1	1
2	3
3	7
4	15
5	31

Claim: $H_n = 2^n - 1$

Pf: We use induction on n. Let $P(n)$ be the statement: "$H_n = 2^n - 1$".

Base case: Using the initial condition, $H_1 = 1 = 2^1 - 1$, so $P(1)$ is true.

Inductive step: Assume $P(k)$ is true: $H_k = 2^k - 1$.
Then using the recurrence relation, we have

$$
\begin{aligned}
H_{k+1} &= 2H_k + 1 && \text{(by the recurrence rel'n)} \\
&= 2(2^k - 1) + 1 && \text{(by the inductive hypothesis} \\
&= 2^{k+1} - 1
\end{aligned}
$$

So $P(k+1)$ is true, and $P(n)$ is true for all n by induction. $\square$

Ex 4: Call a decimal string a "valid codeword" if it has an even num. of 0's. Let a_n be the number of valid codewords of length n. Find a recurrence for a_n.

Sol'n: If $n \geq 2$, to get a valid codeword of length n, either:

- add a non-0 to the end of a valid codeword of length $n-1$

(9 possible last digits) $\cdot$ (a_{n-1} valid codewords) $= 9a_{n-1}$

- or add a 0 to the end of a invalid codeword of length $n-1$

(1 possible last digit) $\cdot$ ($10^{n-1} - a_{n-1}$ valid codewords) $= 10^{n-1} - a_{n-1}$

So $$a_n = 9a_{n-1} + 10^{n-1} - a_{n-1} = 10^{n-1} - 8a_{n-1}$$

Ex 5 (if time): "Catalan numbers"

Let C_n be the number of ways to write n A's and n B's such that as you read left to right, you've never seen more B's than A's

Call this the "catalan property"

e.g. AAABBB valid   BA invalid

ABABAB valid   ABBAAB invalid

Find a recurrence rel'n for C_n.

Sol'n : $C_0 = C_1 = 1$

If $n \geq 1$, consider a sequence w/ $n+1$ A's and $n+1$ B's. Let $k+1$ be the num. A's and B's encountered when you <u>first</u> have the same num of A's and B's

e.g $\underbrace{AB}_{k+1=1}ABAB$ $\qquad \underbrace{AAABBB}_{k+1=3}$

Notice the right part of the string has $n-k$ A's and $n-k$ B's and satisfies the Catalan property; thus, there are C_{n-k} ways to choose it.

$$\underbrace{AB}_{k+1=1}\underbrace{ABAB}_{C_{n-k}=C_2} \qquad \underbrace{AAABBB}_{k+1=3} \quad C_{n-k}=C_0$$

The left part also satisfies the Catalan property, but there's more: the left part always starts w/ an A, ends w/ a B, and if you remove those entries, it <u>still</u> satisfies the Catalan property (otherwise, k would be different). Thus, there are C_k ways to choose the left part.

$$\underbrace{AB}ABAB \qquad \underbrace{AAABBB}$$

$$\underbrace{\cancel{A}\cancel{B}}_{C_k=C_0} \qquad \cancel{A}AABB\cancel{B}$$

$$\underbrace{AABB}_{C_k=C_2}$$

Therefore, we obtain the recurrence relation

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_{n-1} C_1 + C_n C_0$$

$$= \sum_{k=0}^{n} C_k C_{n-k}$$