Quiz today!

Midterm course feedback: very few people have filled out so far

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Recall:

Bayes' Theorem: Assume $p(E), p(F) > 0$. Then,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)} = \frac{p(E|F)\,p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

Ex 2: Suppose one person in 100,000 has a particular rare disease. There is a diagnostic test which is correct

- 99% of the time, when given to a person w/ the disease
- 99.5% of the time, when given to a person w/out the disease

Find the probability that a person who tests positive actually has the disease.

Sol'n: E: tests positive, F: has the disease

Want: $p(F|E)$.

$p(F) = \frac{1}{100000} = 0.00001$ $\quad p(\overline{F}) = 0.99999$

$p(E|F) = 0.99$ $\quad p(E|\overline{F}) = 1 - 0.995 = 0.005$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)} = \frac{p(E|F)\,p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

$$= \frac{0.99 \cdot 0.00001}{0.99 \cdot 0.00001 + 0.005 \cdot 0.99999} = 0.002 = 0.2\%$$

Even though the test is very good, almost all of the positive tests are false positives!

Ex 1 (Class activity if time):

Two boxes

Box 1: 2 Green balls, 7 Red balls

Box 2: 4 Green balls, 3 Red balls

We

- Choose a box at random ($p(\text{Box } 1) = 0.5$)
- Choose a ball at random from that box (equal prob for each ball in the box)

If we select a Red ball, what is the probability it came from the first box?

Sol'n: E: Red ball $\quad \overline{E}$: Green ball

F: Box 1 $\quad \overline{F}$: Box 2

Want: $p(F|E)$

$p(E|F) = \frac{7}{9} \quad p(E|\overline{F}) = \frac{3}{7}$

$p(F) = p(\overline{F}) = \frac{1}{2}$

$$p(E) = p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F}) = \frac{7}{9}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{2} = \frac{38}{63}$$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)} = \frac{\frac{7}{9}\cdot\frac{1}{2}}{38/63} = \frac{49}{76} \approx 0.645$$

§8.1: Recurrence Relations:

Def: A <u>sequence</u> is an infinite list of numbers

$$a_1, a_2, a_3, \ldots$$

doesn't need to start w/ a_1

A <u>recurrence relation</u> is a formula for a_n in terms of (some of) $a_1, a_2, \ldots, a_{n-1}$.

Given a recurrence rel'n and some <u>initial condition(s)</u> (value of at least a_1) we try to <u>solve</u> the recurrence rel'n by giving an explicit formula (not a recurrence rel'n) for a_n.

Ex 1: Fibonacci sequence: $\{f_n\}$

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

$$\left.\begin{array}{l} \text{Recurrence rel'n: } f_n = f_{n-1} + f_{n-2} \\ \text{Initial conds.: } f_1 = 1,\ f_2 = 1 \end{array}\right\} \text{hard to solve}$$

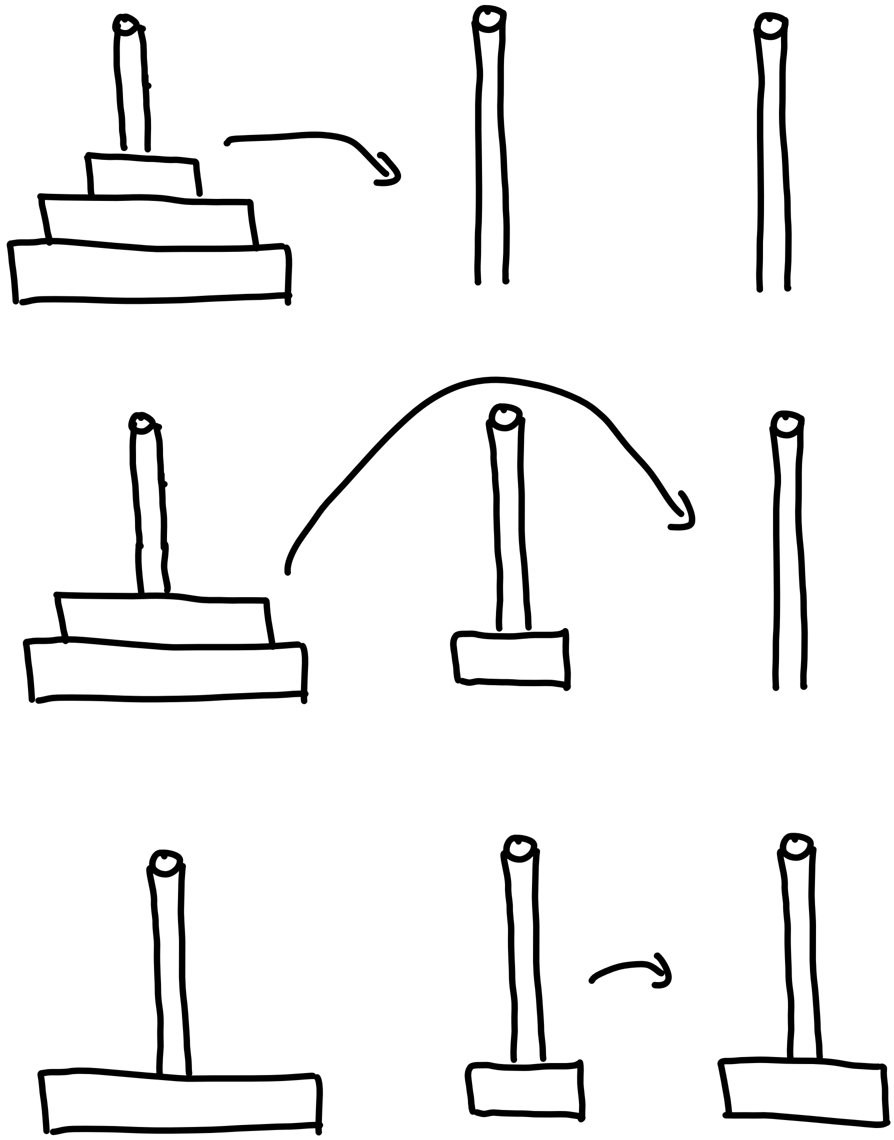
Ex 2: Towers of Hanoi:

3 pegs

n discs of different sizes on Post 1

Want to move them all to Post 2

Can only stack smaller on larger

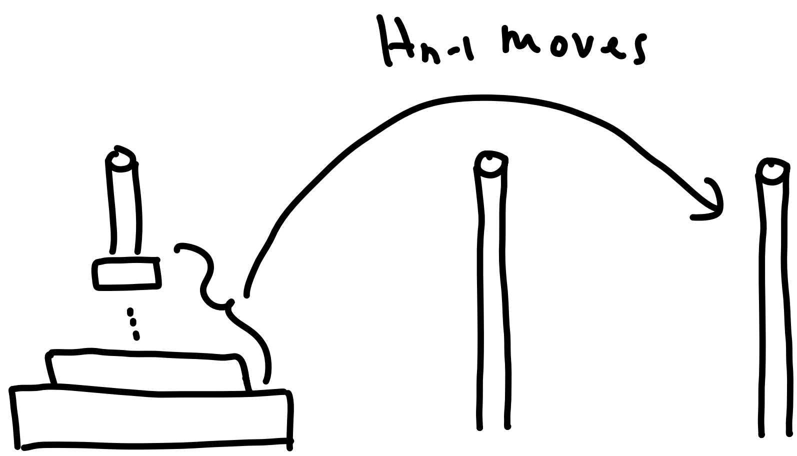


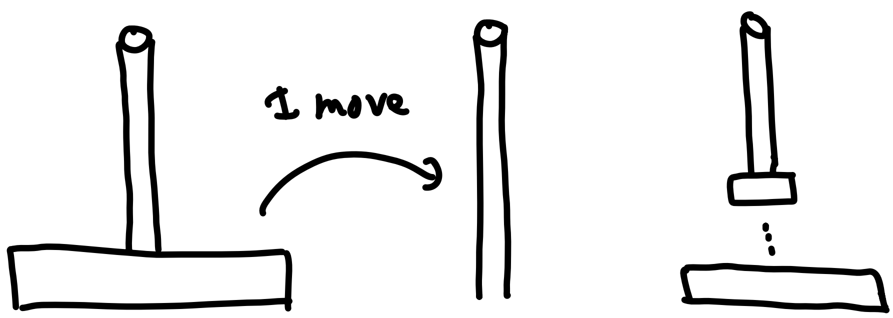
Class activity:

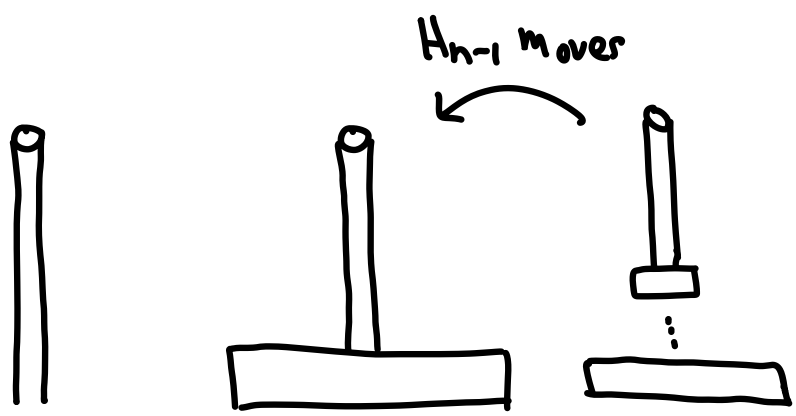
Find the minimum number of moves to move all 3 discs from Post 1 to Post 2

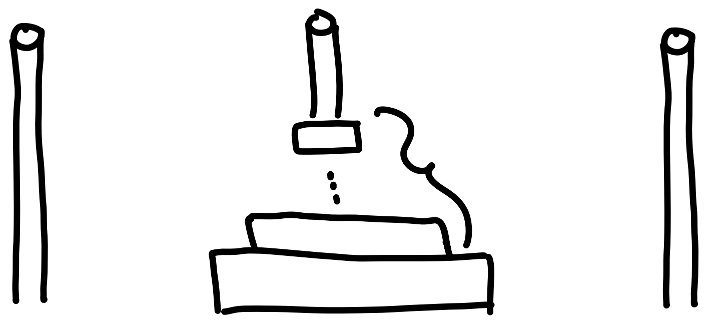
Let H_n be the min. num. moves to move all discs from Post 1 $\to$ Post 2

Now let's find a recurrence for H_n









Recurrence rel'n: $H_n = 2H_{n-1} + 1$

Initial cond.: $H_1 = 1$

n	H_n
1	1
2	3
3	7
4	15
5	31

$H_n = 2^n - 1$

Pf: We use induction on n. Let $P(n)$ be the statement:
"$H_n = 2^n - 1$".

Base case: Using the initial condition,
$H_1 = 1 = 2^1 - 1$, so $P(1)$ is true.

Inductive step: Assume $P(k)$ is true: $H_k = 2^k - 1$.

Then using the recurrence relation, we have

$$H_{k+1} = 2H_k + 1 \quad \text{(by the recurrence rel'n)}$$

$$= 2(2^k - 1) + 1 \quad \text{(by the inductive hypothesis}$$

$$= 2^{k+1} - 1$$

So $P(k+1)$ is true, and $P(n)$ is true for all n by induction. $\square$