Announcements

HW6 posted (due Wed. 3/11)

Quiz 3 Wed. in-class

Midterm feedback form: fill out if you want

Recall: every outcome has a probability $p(s)$

$0 \le p(s) \le 1$ for all $s \in S$ $\qquad$ $\sum_{s \in S} p(s) = 1$

add up $p(s)$ for every elt. $s \in S$

If E is an event,

$$p(E) = \sum_{s \in E} p(s)$$

The <u>conditional probability</u> of E given F is

$$p(E|F) := \frac{p(E \cap F)}{p(F)}$$

Independence: E and F are <u>independent</u> if and only if

$$p(E \cap F) = p(E)\,p(F)$$

$E_1, \ldots, E_n$ are pairwise independent if for all i, j,

$$p(E_i \cap E_j) = p(E_i)\,p(E_j)$$

$E_1, \ldots, E_n$ are (mutually) independent if every eqn

$$p(E_{i_1} \cap \cdots \cap E_{i_k}) = p(E_{i_1}) \cdots p(E_{i_k})$$

holds.

e.g. $p(E_1 \cap E_4 \cap E_6) = p(E_1)\,p(E_4)\,p(E_6)$

[Note: $p(E \cap F) = p(E)\,p(F)$ is equiv. to

$p(E|F) = p(E)$

since $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$]

Ex (cont.) Flip 3 coins

E: at least one head

F: at least one tail

$p(E) = p(F) = \dfrac{7}{8}$

$p(E \cap F) = \dfrac{6}{8}$

$p(E)\,p(F) = \dfrac{7}{8} \cdot \dfrac{7}{8} = \dfrac{49}{64} \neq \dfrac{6}{8}$

E and F are not independent

Bernoulli trials: successive independent weighted coin flips.

$p(\text{success}) = p$

$p(\text{failure}) = q = 1-p$

Think: coin flips w/ $P(H) = p$

Class activity: Flip 3 coins w/ $P(H) = \frac{2}{3}$. Find the prob of

a) no H

b) exactly one H

c) exactly two H's

d) three H's

General formula: n Bernoulli trials. The probability of exactly k successes is:

$$\binom{n}{k} p^k q^{n-k}$$

$\binom{n}{k}$ orders

k successes

n-k failures

Also called the <u>binomial distribution</u>

Ex 9: Generate a binary string where each digit is generated independently and has a 0.9 chance of being a 0. What is the probability that the string has exactly 8 0's?

Ans: $p(\text{exactly } 8\ 0\text{'s}) = \binom{10}{8} 0.9^8 \cdot 0.1^2 = 0.1937$

§7.3: Bayes Theorem

Recall: conditional probability:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Bayes' Theorem: Assume $p(E), p(F) > 0$. Then,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}$$

Pf: By definition,

$$p(E|F) = \frac{p(E \cap F)}{p(F)} \quad \text{and} \quad p(F|E) = \frac{p(E \cap F)}{p(E)},$$

so $p(E \cap F) = p(E|F)\,p(F)$, and combining this w/ the 2nd eqn,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}.$$

$\square$

Important note: often, $p(E)$ is unknown, so Rosen gives the alternate version:

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F})}$$

This is equivalent to our version, since

$$p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F}) = p(E \cap F) + p(E \cap \overline{F}) = p(E)$$

sum rule, since $E \cap F$ and $E \cap \overline{F}$ have no overlap

Ex 2: Suppose one person in 100,000 has a particular rare disease. There is a diagnostic test which is correct

- 99% of the time, when given to a person w/ the disease
- 99.5% of the time, when given to a person w/out the disease

Find the probability that a person who tests positive actually has the disease.

Sol'n: E: tests positive, F: has the disease

Want: $p(F|E)$.

$p(F) = \frac{1}{100000} = 0.00001$ $\quad p(\overline{F}) = 0.99999$

$p(E|F) = 0.99$ $\quad p(E|\overline{F}) = 1 - 0.995 = 0.005$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)} = \frac{p(E|F)\,p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

$$= \frac{0.99 \cdot 0.00001}{0.99 \cdot 0.00001 + 0.005 \cdot 0.99999} = 0.002 = 0.2\%$$

Even though the test is very good, almost all of the positive tests are false positives!

Ex 1 (Class activity if time):

Two boxes

Box 1: 2 Green balls, 7 Red balls

Box 2: 4 Green balls, 3 Red balls

We

- Choose a box at random ($p(\text{Box } 1) = 0.5$)
- Choose a ball at random from that box (equal prob for each ball in the box)

If we select a Red ball, what is the probability it came from the first box?

Sol'n: E: Red ball $\quad \overline{E}$: Green ball

F: Box 1 $\quad \overline{F}$: Box 2

Want: $p(F|E)$

$p(E|F) = \frac{7}{9} \quad p(E|\overline{F}) = \frac{3}{7}$

$p(F) = p(\overline{F}) = \frac{1}{2}$

$$p(E) = p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F}) = \frac{7}{9}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{2} = \frac{38}{63}$$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)} = \frac{\frac{7}{9}\cdot\frac{1}{2}}{38/63} = \frac{49}{76} \approx 0.645$$