§7.1: Discrete Probability

Sample space: set of possible outcomes

Event: a subset of the sample space

e.g. Flip two coins

Sample space: $\{HH, HT, TH, TT\}$

Some events:

Exactly one head: $\{HT, TH\}$

At least one head: $\{HT, TH, HH\}$

Exactly two heads: $\{HH\}$

Exactly three heads: $\emptyset$

If all outcomes in the sample space S are <u>equally likely</u>, and E is an event, then the probability of E is

$$p(E) = \frac{|E|}{|S|} \qquad \text{Always } 0 \le p(E) \le 1$$

Continue the example:

$$S = \{HH, HT, TH, TT\}$$

Exactly one head: $E = \{HT, TH\}$ $\quad p(E) = \frac{2}{4}$

At least one head: $E = \{HT, TH, HH\}$ $\quad p(E) = 3/4$

Exactly two heads: $E = \{HH\}$ $\quad p(E) = 1/4$

Exactly three heads: $E = \emptyset$ $\quad p(E) = 0/4$

Our main tools for these problems are the counting techniques from the last chapter

Ex 2: What is the probability that when two dice are rolled that their sum is exactly 7?

Sample space:

$S = \{\text{all pairs of rolls}\}$

$= \{(1,1),(1,2),\ldots,(2,1),(2,2),\ldots,(6,6)\}$

6 possibilities for each roll
So by the product rule,
$|S| = 6 \cdot 6 = 36$

Event:

$E = \{\text{rolls that add to } 7\}$

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$|E| = 6$

So $p(E) = \frac{|E|}{|S|} = \frac{6}{36}$

Ex 5: Find the probability that a 5-card hand contains 4 cards of one kind

Sol'n: Sample space: $S = \{\text{5-card hands}\}$

$$|S| = \binom{52}{5} = \frac{52!}{5!\,47!} = 2598960$$

Event: $E = \{\text{Hands w/ 4-of-a-kind}\}$

To choose an element of E:

- Choose the kind for the 4-of-a-kind: $\binom{13}{1}$ ways
- Choose 4 cards of this kind: $\binom{4}{4}$ ways
- Choose the last card: $\binom{48}{1}$ ways

By prod. rule,

$$|E| = \binom{13}{1}\binom{4}{4}\binom{48}{1} = 13 \cdot 48$$

$$P(E) = \frac{|E|}{|S|} = \frac{13 \cdot 48}{\left(\frac{52!}{5!\,47!}\right)} = 0.00024$$

Ex 6: What is the probability that a poker hand contains 3 of one kind, 2 of another kind (full house)?

Sol'n: Sample space: $S = \{\text{5-card hands}\}$

$$|S| = \binom{52}{5} = \frac{52!}{5!\,47!}$$

Event: $E = \{\text{Hands w/ full house}\}$

To choose an element of E:

- Choose the kind for the 3-of-a-kind: $\binom{13}{1}$ ways
- Choose 3 cards of this kind: $\binom{4}{3}$ ways
- Choose the kind for the pair: $\binom{12}{1}$ ways
- Choose 2 cards of this kind: $\binom{4}{2}$ ways

By prod. rule,

$$|E| = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

$$P(E) = \frac{|E|}{|S|} = \frac{3744}{2598960} = 0.0014$$

---

§7.2: Probability Theory

Now we make one small change. Instead of every outcome s being equally likely, they now have an individual prob. $p(s)$

$0 \le p(s) \le 1$ for all $s \in S$ $\qquad \sum_{s \in S} p(s) = 1$

Note: $p: S \to [0,1]$ is a function, called a <u>probability distribution</u>.

Everything we do in this section also holds for the equally-likely situation, which is just $\underbrace{p(s) = \frac{1}{|S|}}_{\text{"uniform distribution"}}$ for all $s \in S$

Ex 2: Roll a die where 3 is twice as likely to be rolled as any other number (which are equally likely)

Class activity: Find the prob. of rolling

a) An odd number
b) An even number
c) An even number or a 3
d) A number 3 or less
e) An odd number or a number 3 or less

Recall the complement $\overline{E} = S \setminus E$ of E

1) Complement rule: $p(\overline{E}) = 1 - p(E)$
2) Subtraction rule: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
3) Sum rule: if E_1 & E_2 disjoint, $p(E_1 \cup E_2) = p(E_1) + p(E_2)$

Ex: 3 coins

$|S| = 8$

E: at least one head

F: at least one tail

$\overline{E}$: no heads

$\overline{F}$: no tails

$$p(\overline{E}) = p(\overline{F}) = \frac{1}{8}$$

so $p(E) = p(F) = 1 - \frac{1}{8} = \frac{7}{8}$

$$p(\overline{E} \cup \overline{F}) = p(\overline{E}) + p(\overline{F}) - p(\overline{E} \cap \overline{F}) = \frac{1}{8} + \frac{1}{8} - 0 = \frac{2}{8}$$

By de Morgan's Laws, $\overline{E \cup F} = \overline{E} \cap \overline{F}$

So $p(E \cup F) = 1 - p(\overline{E \cup F}) = 1 - p(\overline{E} \cap \overline{F}) = 1$

$$p(E \cap F) = p(E) + p(F) - p(E \cup F) = \frac{7}{8} + \frac{7}{8} - 1 = \frac{6}{8}$$

Let E, F be events, $p(F) > 0$. The *conditional probability* of E given F is

$$p(E \mid F) := \frac{p(E \cap F)}{p(F)}$$

Basic idea: If we *know* F is true, what is the chance E is true

Ex (cont.):

$$p(E \mid F) = \frac{6/8}{7/8} = \frac{6}{7} \qquad p(F \mid E) = \text{same}$$

$$p(E \mid \overline{E}) = \frac{0}{1/8} = 0$$

$$p(E \mid \overline{F}) = \frac{p(E \cap \overline{F})}{p(\overline{F})} = \frac{?}{7/8} = 1$$

If $\overline{F}$ true,
E always true