Announcements

No quiz Wed. (will be next Wed.)
HW5 posted (due Wed. 3/4)

Recall: Permutations and combinations

$$P(n,k) = \frac{n!}{(n-k)!} \qquad C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

n permute k — n choose k

binomial coefficients

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§6.4: Binomial coeffs. and identities

Class activity:

a) How many 3-elt subsets of $\{A, B, C, D, E, F\}$ are there?

b) How many contain A?

c) How many don't contain A?

d) Can you express the above 3 quantities as binom. coeffs. $C(n,r)$?

e) Does the above say anything about those binom. coeffs.?

Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\iddots & & & \vdots & & & \ddots
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\iddots & & & & & \vdots & & & & & \ddots
\end{array}$$

Pascal's triangle!

Next, let's expand $(x+y)^3$:

$$\begin{aligned}
(x+y)(x+y)(x+y) &= xxx + xxy + xyx + xyy \\
&\quad + yxx + yxy + yyx + yyy \\
&= \underset{\binom{3}{3}}{1x^3} + \underset{\binom{3}{2}}{3x^2y} + \underset{\binom{3}{1}}{3xy^2} + \underset{\binom{3}{0}}{1y^3}
\end{aligned}$$

Binomial theorem:

$$\begin{aligned}
(x+y)^n &= x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + y^n \\
&= \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}
\end{aligned}$$

Ex 4: What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?

Sol'n: $$(2x-3y)^{25} = \sum_{j=0}^{n} \binom{n}{j} (2x)^j (-3y)^{n-j}$$

$$= \cdots + \binom{25}{12} (2x)^{12} (-3y)^{13} + \cdots$$

So the coefficient is

$$\binom{25}{12} 2^{12} \cdot (-3)^{13} = -\binom{25}{12} 2^{12} \cdot 3^{13}$$

We get a surprising number of identities:

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j}$$

$$3^n = (2+1)^n = \sum_{j=0}^{n} 2^j \binom{n}{j}$$

$$0 = (-1+1)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j$$

$$\left(\text{so } \binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots\right)$$

We can also use counting arguments to prove facts about binom. coeffs.

Vandermonde's Identity:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Pf: Suppose we have two disjoint sets A and B, with $|A| = m$, $|B| = n$. We choose r elts. from $A \cup B$.

Method 1: Simply choose r elts. from $A \cup B$. Total num. ways: $\binom{m+n}{r}$

Method 2:
- Choose an integer k, $0 \leq k \leq r$
- Choose $r-k$ elts. from A
- Choose k elts. from B

Total num. ways: $\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$

Since both methods give all possible ways (w/ no overlap) of choosing r elts. from $A \cup B$, we have

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

$\square$

Ex: $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$

Pf: We count the number of binary strings of length $n+1$ w/ $r+1$ 1's in two ways.

Method 1: Choose the $r+1$ positions for the 1's.

Total num. ways: $\binom{n+1}{r+1}$

Method 2:
- Choose the rightmost position for a 1. Call this $j+1$ (must be $\geq r+1$)
- There are no 1's to the right of position $j+1$, but to the left there may be any combination of 0's and 1's. Choose the r remaining 1's out of these j positions.

Total num. ways: $\sum_{j=r}^{n} \binom{j}{r}$

Since both methods give all possible ways (w/ no overlap), we have $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$.

$\square$

§6.5: Generalized Permatation and Combinations

$P(n,k)$ and $\binom{n}{k}$ refer to permutations/combinations <u>without</u> repetition and with <u>distinguishable</u> objects

e.g. ABCDEF

BCAE — 4-permutation

$\{C, E\}$ — 2-combination

If we allow repetition, we allow examples like

BBBB — 4-perm w/ rep.

$\{C, C\}$ — 2-comb. w/ rep.

For a set of size n, the number of

r-perms w/ repetition is

$$n^r$$

(by prod. rule)

r-combs w/ repetition is

$$\binom{n+r-1}{r} \quad (*)$$

Idea behind $(*)$: "sticks and stones".

Stones are the elements

Sticks are the "separators"

Ex 4: 4 different kinds of cookie. How many different ways are there to choose 6 cookies <u>with</u> (potential) repetition?

e.g.

$$* * | * | | * * *$$

2 of type 1   1 of type 2   0 of type 3   3 of type 4

6 stars (cookies)

4 types, so $4-1=3$ separators

Choose the spots for the 6 cookies (or 3 separators)

Num ways:

$$\binom{6+3}{6} = \binom{9}{6} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

Ex 5: How many nonnegative integer sol'ns does the eqn. $x_1 + x_2 + x_3 = 11$ have

11 "stones" $3-1=2$ "sticks"

$$* * | * * * | * * * * * *$$

$$2 + 3 + 6 = 11$$

Num ways: $\binom{11+(3-1)}{11} = \binom{13}{11} = 78$