Announcements

Midterm 1 Wed. in class (see policy email)

Monday's class will be review

AH no office hour today (instead will be Tues. 10:30-11:30 am via Zoom)

Last time: product rule

Today: continue w/ counting rules

Sum rule: If a task can be done either in one of m ways or one of n ways, with no overlap, then there are $m+n$ ways to do the task.

Ex: How many length-2 "words" are there, where the first letter is capital or lower-case, and the second is lower-case?

First letter: $26 + 26 = 52$ choices (sum rule)

Second letter: 26 choices

Total: $52 \cdot 26 = 1352$ "words"

(product rule)

Ex 16: How many passwords are there satisfying:

a) Length 6, 7, or 8

b) Made up of digits and uppercase letters

c) At least one digit

Length 6:

$26 + 10 = 36$ choices for each digit

Total passwords satisfying b):

$$\underbrace{36}_{\substack{\text{1st}\\\text{digit}}} \cdot \underbrace{36}_{\substack{\text{2nd}\\\text{digit}}} \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$$

Passwords containing only letters (i.e. violating c)):

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$$

Length-6 valid passwords: $36^6 - 26^6 = 1{,}867{,}866{,}560$

Length-7 valid passwords: $36^7 - 26^7$

Length-8 valid passwords: $36^8 - 26^8$

Total: $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8 = 2{,}684{,}483{,}063{,}360$

Subtraction rule: If a task can be done either in one of m ways or one of n ways, with overlap of k, then there are m+n-k ways to do the task.

$$|A \cup B| = \underset{m}{|A|} + \underset{n}{|B|} - \underset{k}{|A \cap B|}$$

Ex 18: How many 01-strings of length 8 either start w/ 1 or end w/ 00?

Start w/ 1:

1 * * * * * * *

$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 128$ choices

End w/ 00

* * * * * * 0 0

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 64$ choices

start w/ 1 AND end w/ 00:

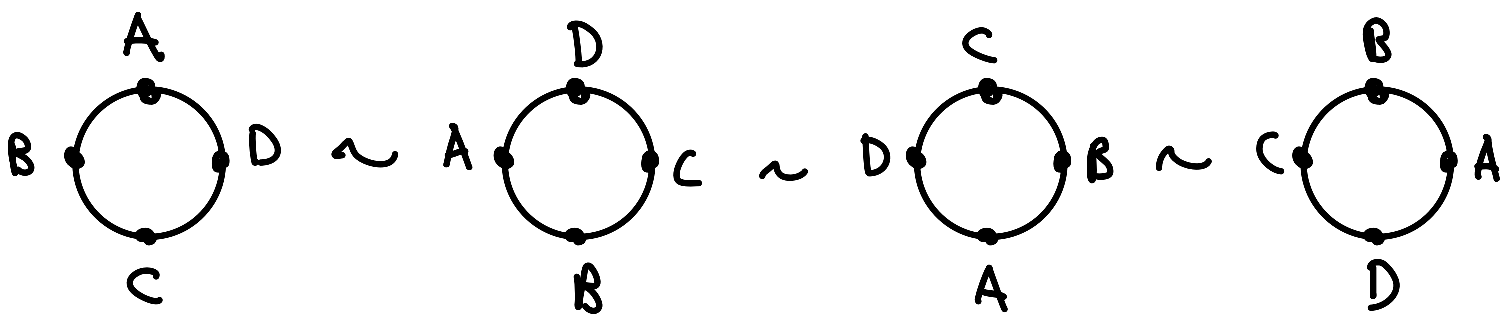
1 * * * * * 0 0

$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 32$ choices

Ans: $128 + 64 - 32 = 160$ strings

Division rule: If there are n ways to do a task, and groups of d of these ways are equivalent, then there are n/d ways up to equivalence.

Ex 20: How many different ways are there to seat 4 people around a circular table, where two seatings are considered equivalent if they are rotations of each other?



4 rotations of each seating arrangement

$4 \cdot 3 \cdot 2 \cdot 1 = 24$ seating arrangements

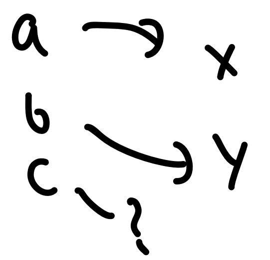
$\frac{24}{4} = 6$ nonequivalent seating arrangements

§6.2 : The Pigeonhole Principle

Pigeonhole Principle: Put m pigeons into n boxes. If $m > n$, there must be at least one box w/ multiple pigeons.

Ex:

a) If $f: A \to B$ and $|A| > |B|$, then f is not 1-1.



b) Among any group of 367 people, there must be at least two who share a birthday

c) For every positive integer n, there is a (nonzero) multiple of n whose base-10 expansion has just 0's and 1's.

E.g. $n = 4$, $4 \cdot 25 = 100$

Class activity: Find such a multiple of 6

Pf: Consider the $n+1$ integers

$$a_1 = 1$$
$$a_2 = 11$$
$$a_3 = 111$$
$$\vdots$$
$$a_{n+1} = \underbrace{11\cdots 1}_{n+1\ 1\text{'s}}$$

Divide each a_i by n, and let r_i be the remainder. Each r_i is an integer from 0 to $n-1$, so by the pigeonhole principle there exist $i<j$ s.t. $r_i = r_j$. Then $n \mid a_j - a_i$ and $a_j - a_i$ has

decimal expansion $\underbrace{1 \cdots 1}_{j-i}\underbrace{0 \cdots 0}_{i}$. $\square$

e.g. $n = 6$

$a_1 = 1$ $\quad r_1 = 1$

$a_2 = 11$ $\quad r_2 = 5$

$a_3 = 111$ $\quad r_3 = 3$

$a_4 = 1111$ $\quad r_4 = 1$

$a_5 = 11111$ $\quad r_5 = 5$

$a_6 = 111111 \quad r_6 = 3$

$a_7 = 1111111 \quad r_7 = 1$

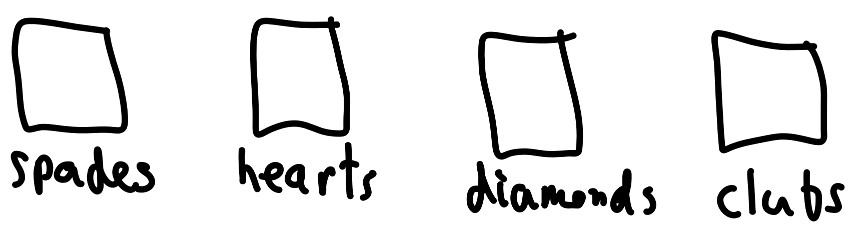
$1111 - 1 = 1110 = 185 \cdot 6.$

Generalized pigeonhole principle: Put m pigeons into n boxes. Then there is at least one box w/ $\lceil m/n \rceil$ pigeons

e.g. $m = 31 \quad n = 10 \rightsquigarrow \lceil m/n \rceil = \lceil 3.1 \rceil = 4$

$m = 40 \quad n = 10 \rightsquigarrow \lceil m/n \rceil = \lceil 4 \rceil = 4$

Ex 7: How many cards must be chosen from a deck to ensure there are ≥ 3 of the same suit



Ans: We want the smallest m s.t.

$\left\lceil \frac{m}{4} \right\rceil \geq 3 \quad$ i.e. $m > 2 \cdot 4 \rightsquigarrow m = 9$

Ex 8: Telephone numbers are of the form,

$$\underbrace{NXX}_{\text{area code}} - NXX - XXXX$$

where each N can be a digit from 2 to 9 and each X can be a digit from 0 to 9.
A state has 25,000,000 phones. How many area codes does it need to ensure each phone has a diff. num?

Sol'n: NXX - XXXX

$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$ numbers per area code

$m = 25$ million, $n = 8$ million

$$\lceil m/n \rceil = \left\lceil \frac{25}{8} \right\rceil = 4$$

So we need 4 area codes.