Problem §2.3: 2: Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

(a) $f(n) = \pm n$

(b) $f(n) = \sqrt{n^2+1}$

(c) $f(n) = \frac{1}{n^2-4}$

Solution. (a) **No**. This is not a function because $f(n)$ is not well-defined, i.e. it does not map each element of the domain to a *single* element of the codomain.

(b) **Yes**. For all $z \in \mathbb{Z}$, the image $f(z) = \sqrt{z^2+1}$ is well-defined and lies in the codomain, $\mathbb{R}$.

(c) **No**, because $f(z)$ is not defined for all $z \in \mathbb{Z}$. Observe that for both $z = 2$ and $z = -2$, $f(z)$ is undefined because it would involve division by zero. In order for $f(n)$ to be a function with domain $\mathbb{Z}$, it would need to be defined on all elements of $\mathbb{Z}$.

$\square$

Problem §2.3: 12: Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

(a) $f(n) = n - 1$.

(b) $f(n) = n^2 + 1$.

(c) $f(n) = n^3$.

(d) $f(n) = \lceil n/2 \rceil$.

Solution. Recall that a function $f : A \to B$ is one-to-one if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for all $a_1, a_2 \in A$. To show that a function is not one-to-one, it is sufficient to find a single counterexample where $a_1 \neq a_2$ but $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$.

Yes, this function is one-to-one. For any $n_1, n_2 \in \mathbb{Z}$, observe that if $f(n_1) = f(n_2)$ then

$$f(n_1) = n_1 - 1 = n_2 - 1 = f(n_2),$$

which implies that $n_1 = n_2$.

(b) **No**, this function is not one-to-one. Observe that, for example,

$$f(-2) = (-2)^2 + 1 = 5 = 2^2 + 1 = f(2),$$

but $-2 \neq 2$.

(c) **Yes**, this function is one-to-one. For any $n_1, n_2 \in \mathbb{Z}$, observe that if $f(n_1) = f(n_2)$ then

$$f(n_1) = n_1^3 = n_2^3 = f(n_2),$$

which implies that $n_1 = n_2$ because all real numbers have a unique cube root.

(d) **No**, this function is not one-to-one. For example, observe that

$$f(3) = \lceil 3/2 \rceil = 2 = \lceil 4/2 \rceil = f(4),$$

but $3 \neq 4$.

$\square$

Problem §2.3: 14(a,b,c,d): Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m, n) = 2m - n$.

(b) $f(m, n) = m^2 - n^2$.

(c) $f(m, n) = m + n + 1$.

(d) $f(m, n) = |m| - |n|$.

Solution. Recall that a function $f : A \to B$ is onto if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. To show that a function is not onto, it's sufficient to find a single $b \in B$ that is not the image of any element of the domain, A.

(a) **Yes**, this function is onto. Observe that any integer z in the codomain, $\mathbb{Z}$, is the image of $(0, -z)$:

$$f(0, -z) = 2(0) - (-z) = z.$$

(b) **No**, this function is not onto. For example, 2 is in its codomain but not its range. Observe that if

$$m^2 - n^2 = (m - n)(m + n) = 2,$$

then m and n must have the same parity, i.e. must both either be even or odd (if m and n had different parities, then both $m - n$ and $m + n$ would be odd, forcing their product, $m^2 - n^2$, to also be odd). If m and n have the same parity, then both $m - n$ and $m + n$ are even and therefore divisible by 2. Hence, their product is divisible by 4 and cannot be equal to 2.

(c) **Yes**, this function is onto. Observe that any integer z in the codomain $\mathbb{Z}$ is the image of $(0, z - 1)$:

$$f(0, z - 1) = 0 + (z - 1) + 1 = z.$$

(d) **Yes**, this function is onto. Observe that any positive integer z in the codomain $\mathbb{Z}$ is the image of $(z, 0)$, any negative integer z is the image of $(0, z)$, and 0 is the image of $(0, 0)$:

$$\begin{aligned} f(z, 0) &= |z| - |0| = |z| = z, && \text{for } z \in \mathbb{Z}_{\geq 0}, \\ f(0, z) &= |0| - |z| = -|z| = -(-z) = z, && \text{for } z \in \mathbb{Z}_{\leq 0}, \\ f(0, 0) &= |0| - |0| = 0. \end{aligned}$$

□

Problem §2.3: 20: Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

(a) one-to-one but not onto.

(b) onto but not one-to-one.

(c) both onto and one-to-one (but not the identity function).

(d) neither one-to-one nor onto.

Solution. (a) The function $f(n) = n + 1$ is one-to-one but not onto. To see that it's one-to-one, observe that for all $n_1, n_2 \in \mathbb{N}$, if $f(n_1) = f(n_2)$ then $n_1 + 1 = n_2 + 1$ which implies $n_1 = n_2$. It's not onto, however, because 0 is not the image of any natural number. To see this, observe that if we had $0 = f(n) = n + 1$, this would require that $n = -1$ and $-1 \notin \mathbb{N}$.

(b) The function $f(n) = \lceil n/2 \rceil$ is onto but not one-to-one. Observe that any element n of the codomain is the image of both $2n$ and $2n + 1$:

$$f(2n) = \lceil (2n)/2 \rceil = \lceil n \rceil = n,$$

$$f(2n + 1) = \lceil \frac{2n + 1}{2} \rceil = \lceil n + 1/2 \rceil = n.$$

(c) Consider the piecewise function

$$f(n) = \begin{cases} n-1 & \text{n even} \\ n+1 & \text{n odd} \end{cases}$$

which "swaps" the even and odd natural numbers. For example, $f(1) = 2$ and $f(2) = 1$, $f(3) = 4$ and $f(4) = 3$, etc. This function is onto because each even n in the codomain is the image of $n-1$ and each odd n in the codomain is the image of $n+1$. It is also one-to-one. To see that it is one-to-one, observe that if $f(n_1) = f(n_2)$, then either $n_1 - 1 = n_2 - 1$ or $n_1 + 1 = n_2 + 1$, depending on parity. In either case, this implies $n_1 = n_2$.

(d) The function $f(n) = 0$ is clearly neither onto nor one-to-one because it maps every element of the domain to the same element of the codomain.

□

Problem §2.3: 22(a,b): Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = -3x + 4$.

(b) $f(x) = -3x^2 + 7$.

Solution. Recall that a bijection is a function that is both injective (one-to-one) and surjective (onto). So one strategy would be to determine if each function is both injective or surjective. To save ourselves some work, though, when we want to show that a function is a bijection we can use the fact that only bijections have inverses. Showing that an inverse function exists is, therefore, equivalent to showing that the function is a bijection.

(a) **Yes**, this function is a bijection. We claim that the inverse function of f is $f^{-1}(x) = (4-x)/3$. To verify this, observe that for $x \in \mathbb{R}$,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(-3x+4) = \frac{4-(-3x+4)}{3} = \frac{3x}{3} = x,$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{4-x}{3}\right) = -3\left(\frac{4-x}{3}\right) + 4 = -4 + x + 4 = x$$

(b) **No**, this function is not a bijection because it's not injective *or* surjective. To see that it's not injective, observe that, for example, $f(-1) = -3(-1)^2 + 7 = 4 = -3(1)^2 + 7 = f(1)$, but $-1 \neq 1$. To see that it's not surjective, observe that $x^2 \geq 0$ for all $x \in \mathbb{R}$. As such, the range of $f(x)$ is $(-\infty, 7]$, which is clearly not equal to the codomain $\mathbb{R}$.

□

Problem §2.3: 36: Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution. Because both f and g are functions from $\mathbb{R}$ to $\mathbb{R}$, the compositions $f \circ g$ and $g \circ f$ are well-defined. We can compute these compositions as:

$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5,$$
$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3.$$

Notice that $g \circ f \neq f \circ g$! □

Problem §2.3: 39: Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

Solution. One easy way to show that the given function f is invertible is to exhibit an inverse function. We claim that it has inverse function

$$\begin{aligned} f^{-1} &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \frac{x-b}{a}. \end{aligned}$$

To verify that this is the inverse function of f, we need to check that both $(f \circ f^{-1})$ and $(f^{-1} \circ f)$ are the identity function on $\mathbb{R}$. We can do so by computing

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) = f\left(\frac{x-b}{a}\right) = a\left(\frac{x-b}{a}\right) + b = x - b + b = x, \\ (f^{-1} \circ f)(x) &= f^{-1}(f(x)) = f^{-1}(ax+b) = \frac{(ax+b)-b}{a} = \frac{ax}{a} = x. \end{aligned}$$

□

Problem §2.3: 40(a): Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that $f(S \cup T) = f(S) \cup f(T)$.

Solution. We'll show that $f(S \cup T) = f(S) \cup f(T)$ by showing that each set is a subset of the other.

First, suppose that $b \in f(S \cup T)$. By definition, this means that $b = f(a)$ for some $a \in S \cup T$. By definition of union, either $a \in S$, $a \in T$, or both. If $a \in S$, then $f(a) \in f(S)$. If $a \in T$, then $f(a) \in f(T)$. Thus, in any case we have $f(a) \in f(S) \cup f(T)$. Hence, $f(S \cup T) \subseteq f(S) \cup f(T)$.

Conversely, suppose that $b \in f(S) \cup f(T)$. Then by definition, $b \in f(S)$ or $b \in f(T)$ or both. If $b \in f(S)$, then by definition $b = f(a)$ for some $a \in S$. Similarly, if $b \in f(T)$ then by definition $b = f(a)$ for some $a \in T$. So in every case, we have $b = f(a)$ for some $a \in S \cup T$ and by definition $b \in f(S \cup T)$.

Since we've shown both inclusions, we have therefore shown that $f(S \cup T) = f(S) \cup f(T)$, as desired. □

Problem §2.3: 44(b): Let f be a function from A to B. Let S and T be subsets of B. Show that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Solution. Again, we'll show the desired set equality by showing that each set is a subset of the other.

First, consider $a \in f^{-1}(S \cap T)$. By definition, $f(a) \in S \cap T$ and therefore either $f(a) \in S$ and $f(a) \in T$. The fact that $f(a) \in S$ means $a \in f^{-1}(S)$. Similarly, the fact that $f(a) \in T$ means that $a \in f^{-1}(T)$. Hence, by definition $a \in f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Conversely, consider $a \in f^{-1}(S) \cap f^{-1}(T)$. By definition, $a \in f^{-1}(S)$ and $a \in f^{-1}(S)$. From the definition of the preimage of a set, we know that $a \in f^{-1}(S)$ means that $f(a) \in S$. Similarly, $a \in f^{-1}(T)$ means that $f(a) \in T$. As such, we have $f(a) \in S \cap T$ and therefore $a \in f^{-1}(S \cap T)$. Thus, $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

Because we've shown both inclusions, we have therefore shown that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, as desired. □