# Solutions to Math 213-A1 Final Exam — May 12, 2026

1. (18 points) Answer the following questions.
*(No work necessary for this problem! Only your answer will be graded.)*
*(For this problem, you may leave your answer in terms of binomial coefficients.)*

   (a) (3 points) Write the negation of the following statement using quantifiers:

$$\forall x\, \exists y\, (xy = 1),$$

   where the domain of both variables is all real numbers.

   $\exists x\, \forall y\, (xy \neq 1)$.

   (b) (3 points) True or false: for any sets $A, B, C$, we must have $(A \setminus B) \times C \subseteq (A \times C) \setminus (B \times C)$.

   True. Proof: if $(a, c) \in (A \setminus B) \times C$, then $a \in A \setminus B$ and $c \in C$. This means that $a \in A, a \notin B$, so $(a, c) \in A \times C$ but $(a, c) \notin B \times C$.
   (In fact, the sets are equal.)

   (c) (3 points) How many permutations of the letters $ABCDEF$ contain either the substring $AB$ or the substring $BA$?

   Since a permutation cannot contain both $AB$ and $BA$ (each letter appears only once), there is no need for inclusion-exclusion. Treating $AB$ as a single block gives 5! permutations containing $AB$, and similarly 5! containing $BA$. The total is $2 \cdot 5! = 240$.

   (d) (3 points) How many ways are there to buy 12 cookies from a shop with 5 flavors, if you can buy as many as you want of each flavor?

   By sticks and stones, there are $\binom{12+5-1}{12} = \binom{16}{12}$ ways.

   (e) (3 points) What is the probability that a random five-card poker hand contains exactly 2 face cards? (Recall that a standard deck has 12 face cards out of 52 total cards.)

   Choose 2 of 12 face cards and 3 of 40 non-face cards:

$$\frac{\binom{12}{2}\binom{40}{3}}{\binom{52}{5}}.$$

   (f) (3 points) Let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$ be a relation on $\{a, b, c\}$. Is $R$ reflexive? Symmetric? Transitive?

   $R$ is reflexive (contains $(a, a), (b, b), (c, c)$) and symmetric ($(a, b) \leftrightarrow (b, a)$, $(b, c) \leftrightarrow (c, b)$). However, it is not transitive: $(a, b) \in R$ and $(b, c) \in R$, but $(a, c) \notin R$.

2. (15 points) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n$, with initial conditions $a_0 = 2$ and $a_1 = 1$.

   This is a linear inhomogeneous recurrence relation, so all solutions are of the form $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(p)}$ is any particular solution and $a_n^{(h)}$ is a solution to the homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

   The characteristic equation is $r^2 - 5r + 6 = (r - 2)(r - 3) = 0$, so there are two roots, 2 and 3. Therefore, (by Theorem 8.2.3) the general solution to the homogeneous equation is

$$a_n^{(h)} = \alpha \cdot 2^n + \beta \cdot 3^n,$$

where $\alpha$ and $\beta$ are arbitrary.

Next, for the particular solution. Since the inhomogeneous part is $F(n) = 2^n$ and 2 *is* a root of the characteristic equation with multiplicity $m = 1$, we know (Theorem 8.2.6) that there is a particular solution of the form $a_n^{(p)} = pn2^n$.

We plug this particular solution into the recurrence relation to obtain

$$pn2^n = 5p(n-1)2^{n-1} - 6p(n-2)2^{n-2} + 2^n.$$

Factoring out $2^{n-2}$, this equation becomes $4pn = 10p(n-1) - 6p(n-2) + 4$. Expanding,

$$4pn = 10pn - 10p - 6pn + 12p + 4 = 4pn + 2p + 4,$$

and solving for $p$ we get $p = -2$. Therefore, the general solution is

$$a_n = \alpha \cdot 2^n + \beta \cdot 3^n - 2n \cdot 2^n.$$

Applying the initial conditions: $a_0 = \alpha + \beta = 2$ and $a_1 = 2\alpha + 3\beta - 4 = 1$, so $2\alpha + 3\beta = 5$. Substituting $\alpha = 2 - \beta$ gives $2(2-\beta) + 3\beta = 5$, so $\beta = 1$ and $\alpha = 1$.

The solution is $a_n = 2^n + 3^n - 2n \cdot 2^n$.

3. (12 points) Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3\}$, and $C = \{3, 4, 5\}$. Write the following sets.
*(No work necessary for this problem! Only your answer will be graded.)*

   (a) (3 points) $A \cap (B \cup C)$

   $B \cup C = \{2, 3, 4, 5\}$, so $A \cap (B \cup C) = \{2, 3, 4\}$.

   (b) (3 points) $\mathcal{P}(B)$

   $\mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$.

   (c) (3 points) $B \times C$

   $B \times C = \{(2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$.

   (d) (3 points) $A \setminus (B \setminus C)$

   $B \setminus C = \{2\}$, so $A \setminus (B \setminus C) = \{1, 3, 4\}$.

4. (14 points) Determine, with justification, whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is injective, surjective, both, or neither.

   (a) (4 points) $f(n) = n^2$

   $f$ is neither injective nor surjective. It is not injective because $f(1) = 1 = f(-1)$, but $1 \neq -1$. It is not surjective because $-1$ is not in the image of $f$: there is no integer $n$ with $n^2 = -1$, since $n^2 \geq 0$ for all $n$.

   (b) (4 points) $f(n) = 2n + 1$

   $f$ is injective but not surjective. It is injective because if $f(x) = f(y)$, then $2x + 1 = 2y + 1$, so $2x = 2y$, so $x = y$. It is not surjective because 0 is not in the image of $f$: $2n + 1$ is always odd, so $f(n) \neq 0$ for any $n$.

   (c) (6 points) Consider the equivalence relation $\equiv$ on $\mathbb{Z}$ defined by $a \equiv b$ if and only if $a - b$ is a multiple of 3. There are three equivalence classes: $[0] = \{\ldots, -3, 0, 3, 6, \ldots\}$, $[1] = \{\ldots, -2, 1, 4, 7, \ldots\}$, and $[-1] = \{\ldots, -1, 2, 5, 8, \ldots\}$. Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n & \text{if } n \in [0], \\ n+1 & \text{if } n \in [1], \\ n-1 & \text{if } n \in [-1]. \end{cases}$$

$f$ is bijective (both injective and surjective).

First, observe that $f$ maps each equivalence class to a different one: $f$ sends $[0]$ to $[0]$ (since $n \in [0]$ gives $f(n) = n \in [0]$), $[1]$ to $[-1]$ (since $n \in [1]$ gives $f(n) = n + 1 \in [-1]$), and $[-1]$ to $[1]$ (since $n \in [-1]$ gives $f(n) = n - 1 \in [1]$).

*Injective:* Suppose $f(x) = f(y)$. Since $f$ maps the three equivalence classes to three *different* equivalence classes, $x$ and $y$ must lie in the same class (otherwise $f(x)$ and $f(y)$ would be in different classes). If $x, y \in [0]$, then $f(x) = x = y = f(y)$, so $x = y$. If $x, y \in [1]$, then $f(x) = x + 1 = y + 1 = f(y)$, so $x = y$. If $x, y \in [-1]$, then $f(x) = x - 1 = y - 1 = f(y)$, so $x = y$.

*Surjective:* Let $z \in \mathbb{Z}$. If $z \in [0]$, then $f(z) = z$. If $z \in [-1]$, then $z - 1 \in [1]$ and $f(z - 1) = (z - 1) + 1 = z$. If $z \in [1]$, then $z + 1 \in [-1]$ and $f(z + 1) = (z + 1) - 1 = z$. In every case, $z$ is in the image of $f$.

5. (12 points) Prove that $f(n) = n^3 \cdot 2^n$ is $O(n!)$.

   *(Your proof must be fully rigorous. You may not assume without proof that one function is eventually greater than another; all intermediate inequalities require justification.)*

Let $C = 1$, $k = 9$.

First, we show that $4^n < n!$ for all $n \geq 9$, by induction.

Base: If $n = 9$, then $4^9 = 262{,}144 < 362{,}880 = 9!$.

Inductive Step: Suppose that $n \geq 9$ and $4^n < n!$. Then

$$4^{n+1} = 4 \cdot 4^n < 4 \cdot n! \leq (n + 1) \cdot n! = (n + 1)!,$$

where the last step uses $n + 1 \geq 10 > 4$. Thus, we have shown that $4^n < n!$ for all $n \geq 9$ by induction.

Next, we show that $n^3 < 2^n$ for all $n \geq 10$, by induction.

Base: If $n = 10$, then $n^3 = 1000 < 1024 = 2^{10}$.

Inductive Step: Suppose that $n \geq 10$ and $n^3 < 2^n$. Then $\frac{n+1}{n} \leq \frac{11}{10}$, so

$$(n + 1)^3 = n^3 \cdot \left(\frac{n + 1}{n}\right)^3 \leq n^3 \cdot \left(\frac{11}{10}\right)^3 = \frac{1331}{1000} n^3 < 2n^3 < 2 \cdot 2^n = 2^{n+1}.$$

Thus, we have shown that $n^3 < 2^n$ for all $n \geq 10$ by induction.

Therefore, if $n > k$,

$$|f(n)| = n^3 \cdot 2^n < 2^n \cdot 2^n = 4^n < n!,$$

so $f$ is $O(n!)$.

6. (10 points) Suppose that 2% of people have a certain disease. There is a test for the disease such that 80% of people with the disease test positive and only 10% without the disease test positive. What is the probability that someone who tests positive has the disease?

Let $E$ be the event of having the disease and $F$ be the event of testing positive. We want to compute $P(E \mid F)$. From the problem statement, $P(E) = 0.02$, $P(\overline{E}) = 0.98$, $P(F \mid E) = 0.8$, and $P(F \mid \overline{E}) = 0.1$. Applying Bayes' Theorem,

$$P(E \mid F) = \frac{P(F \mid E)\,P(E)}{P(F \mid E)\,P(E) + P(F \mid \overline{E})\,P(\overline{E})} = \frac{0.8 \cdot 0.02}{0.8 \cdot 0.02 + 0.1 \cdot 0.98} = \frac{0.016}{0.114} \approx 14\%.$$

(Since no calculators are allowed, $\frac{0.8\cdot 0.02}{0.8\cdot 0.02+0.1\cdot 0.98}$ is an acceptable final answer.)

7. (15 points) Let $a_n$ be the number of ways to tile a $1 \times n$ board using tiles of three types:

- Red tiles (R) of length 1,
- Blue tiles (B) of length 1, and
- Green tiles (G) of length 2.

For example, $a_3 = 12$, since the valid tilings of a $1 \times 3$ board are RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB, RG, BG, GR, GB.

(a) (5 points) Find a recurrence relation for $a_n$.

Consider the last tile placed. If it is a red or blue tile (length 1), there are $a_{n-1}$ ways to tile the remaining $1 \times (n-1)$ board, and there are 2 color choices. If it is a green tile (length 2), there are $a_{n-2}$ ways to tile the remaining $1 \times (n-2)$ board. Therefore,

$$a_n = 2a_{n-1} + a_{n-2}, \qquad n \geq 2.$$

(b) (5 points) What are the initial conditions?

An empty board can be tiled in 1 way (leave it empty), so $a_0 = 1$. A board of length 1 can be tiled with R or B, so $a_1 = 2$.

(c) (5 points) Compute $a_6$ using your recurrence relation and initial conditions.

$$\begin{aligned} a_2 &= 2(2) + 1 = 5 \\ a_3 &= 2(5) + 2 = 12 \\ a_4 &= 2(12) + 5 = 29 \\ a_5 &= 2(29) + 12 = 70 \\ a_6 &= 2(70) + 29 = 169 \end{aligned}$$

8. (15 points) Let $n$ and $k$ be positive integers. **Using a combinatorial argument**, prove the following identity:

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k}\binom{n-j}{k}$$

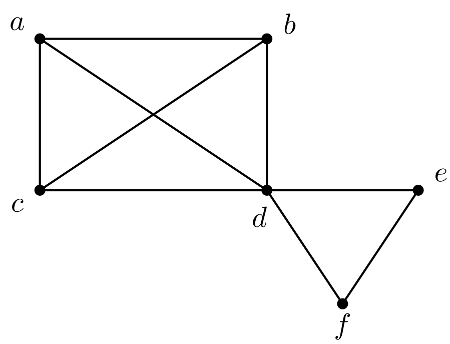
We count the number of binary strings of length $n+1$ with $2k+1$ 1's and $n-2k$ 0's. We do this in two ways. On one hand, choosing the positions of the 1's clearly gives $\binom{n+1}{2k+1}$.

On the other hand, let $j+1$ be the position of the *middle* 1, i.e. the $(k+1)$st 1 reading from left to right. There are $k$ 1's both before and after position $j+1$, so we must have $k < j+1 \leq n+1-k$, so $k \le j \le n-k$. Then we can (independently) choose the $k$ 1's before position $j+1$, and the $k$ 1's after position $j+1$. The former choice has $\binom{j}{k}$ possibilities, while the latter has $\binom{n-j}{k}$ possibilities. Since we are counting the same set in multiple ways, we have

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k}\binom{n-j}{k},$$

as desired.

9. (10 points) Determine the chromatic number of the following graph. *(You must explain why the number you found is both a lower bound and an upper bound for the chromatic number.)*




The chromatic number is $\chi(G) = 4$.

*Lower bound:* The vertices $a$, $b$, $c$, and $d$ are mutually adjacent (edges $ab$, $ac$, $ad$, $bc$, $bd$, and $cd$), forming a $K_4$. Any $K_4$ requires 4 distinct colors, so $\chi(G) \ge 4$.

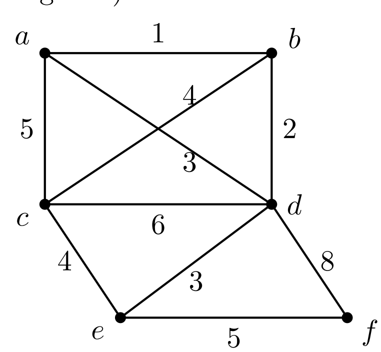
*Upper bound:* Here is a valid 4-coloring: $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 1$, $f = 2$. We verify:

$$ab\,(1,2),\ ac\,(1,3),\ ad\,(1,4),\ bc\,(2,3),\ bd\,(2,4),\ ce\,(3,1),\ de\,(4,1),\ df\,(4,2),\ ef\,(1,2).$$

All pairs of adjacent vertices have different colors, so this is a proper 4-coloring.

Combining, $\chi(G) = 4$.

10. (12 points) (a) (6 points) Use Kruskal's algorithm to find a minimal spanning tree for the following weighted graph. List the edges in the order they are added, and give the total weight (sum of the weights for the edges in the spanning tree).




Sort edges by weight: $ab = 1,\ bd = 2,\ ad = 3,\ de = 3,\ bc = 4,\ ce = 4,\ ac = 5,\ ef = 5,\ cd = 6,\ df = 8$.

Apply Kruskal's algorithm:

1. $ab = 1$: add.
2. $bd = 2$: add.
3. $ad = 3$: **skip** (creates cycle $a–b–d–a$).
4. $de = 3$: add.
5. $bc = 4$: add.
6. $ef = 5$: add. All vertices connected — done.

The minimal spanning tree consists of edges $ab, bd, de, bc, ef$ with total weight $1+2+3+4+5 = 15$.

(b) (6 points) Use the algorithm discussed in class to construct a binary search tree from the following list:

Thank, You, For, Your, Hard, Work, Learning, Discrete, Mathematics,

inputting the words in the order given, and using alphabetical order. *(Note: "you" comes before "your" in alphabetical order).*

Alphabetical order: Discrete < For < Hard < Learning < Mathematics < Thank < Work < You < Your.

We insert each word in order:

- **Thank**: root.
- **You**: You > Thank, so right child of Thank.
- **For**: For < Thank, so left child of Thank.
- **Your**: > Thank, > You, so right child of You.
- **Hard**: < Thank, > For, so right child of For.
- **Work**: > Thank, < You, so left child of You.
- **Learning**: < Thank, > For, > Hard, so right child of Hard.
- **Discrete**: < Thank, < For, so left child of For.
- **Mathematics**: < Thank, > For, > Hard, > Learning, so right child of Learning.