Note: the distribution of these problems may not match the distribution of exam topics.

Problem §2.3: 34: If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Solution. Yes. Suppose that $g(x) = g(y) = b$. Then

$$(f \circ g)(x) = f(g(x)) = f(b) = f(g(y)) = (f \circ g)(y),$$

and since $f \circ g$ is one-to-one, $x = y$. □

Problem §2.2: 30: Can you conclude that $A = B$ if A, B, and C are sets such that

(a) $A \cup C = B \cup C$

(b) $A \cap C = B \cap C$

(c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$

Solution. (a) No. Let $A = \{1\}, B = \{2\}, C = \{1, 2\}$. Then $A \cup C = \{1, 2\} = A \cup B$, but $A \neq B$.

(b) No. Let $A = \{1\}, B = \{2\}, C = \emptyset$. Then $A \cup C = \emptyset = A \cup B$ but $A \neq B$.

(c) Yes. If $x \in A$, then either $x \in C$ or $x \notin C$. In the first case, $x \in A \cap C = B \cap C$, so $x \in B$. In the second case, $x \in A \cup C = B \cup C$, so since $x \notin C, x \in B$. Therefore $A \subseteq B$, and a similar argument shows that $B \subseteq A$.

□

Problem §2.3: 34: If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Solution. Yes. Suppose that $g(x) = g(y) = b$. Then

$$(f \circ g)(x) = f(g(x)) = f(b) = f(g(y)) = (f \circ g)(y),$$

and since $f \circ g$ is one-to-one, $x = y$. □

Problem §3.2: 18: Let k be a positive integer. Show that $f(k) = 1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$.

Solution. Let $C = 1$ and $K = 1$ (we use K instead of the usual k since k is already used in the problem statement). Then for $x > 1$, we have

$$|f(k)| = 1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots n^k = n(n^k) = n^{k+1} = C|n^{k+1}|,$$

so $f(k)$ is $O(n^{k+1})$. □

Problem §5.1: 12: Prove that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

whenever n is a nonnegative integer.

Solution. We'll proceed by mathematical induction.

Base Case: When $n = 0$, observe that

$$\left(-\frac{1}{2}\right)^0 = 1,$$
$$\frac{2^{0+1} + (-1)^0}{3 \cdot 2^0} = \frac{2+1}{3} = 1,$$

so $P(1)$ is the trivially true statement that $1 = 1$.

Inductive Step: Assume, for some $k \in \mathbb{Z}_{>0}$, that

$$\sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k}.$$

We can then observe that

$$\begin{aligned}
\sum_{j=0}^{k+1} \left(-\frac{1}{2}\right)^j &= \left(-\frac{1}{2}\right)^{k+1} + \sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j \\
&= \frac{(-1)^{k+1}}{2^{k+1}} + \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} \qquad \text{(by the IHOP)} \\
&= \frac{3(-1)^{k+1} + 2^{k+2} + 2(-1)^k}{3 \cdot 2^{k+1}} \\
&= \frac{3(-1)^{k+1} + 2^{k+2} + (-2)(-1)^{k+1}}{3 \cdot 2^{k+1}} \\
&= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}.
\end{aligned}$$

Conclusion: Because we showed that the formula holds for $n = 1$ and that holding for $n = k$ implies it also holds for $n = k + 1$, we have shown by the principle of mathematical induction that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

whenever n is a nonnegative integer. □

Problem §5.1: 24: Prove that

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}$$

whenever n is a positive integer.

Solution. We wish to show that

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}$$

for all positive integers n.

Base Case: When $n = 1$, this is simply the statement that

$$\frac{1}{2} \leq \frac{1}{2},$$

which is clearly true.

Inductive Step: Assume that

$$\frac{1}{2k} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots 2k}$$

for some positive integer k. Then observe that

$$\begin{aligned} \frac{1}{2(k+1)} &= \frac{1}{2k} \cdot \frac{2k}{2(k+1)} \\ &\leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots 2k} \cdot \frac{2k}{2(k+1)} \qquad \text{(by the IHOP)} \\ &\leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdots 2k} \cdot \frac{2k+1}{2(k+1)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2(k+1)-1)}{2 \cdot 4 \cdots 2k \cdot 2(k+1)}. \end{aligned}$$

Conclusion: Because we verified the base case $P(1)$ and showed that $P(k)$ implies $P(k+1)$ for all $k \geq 1$, we know by the principle of mathematical induction that for all $n \geq 1$,

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}.$$

□

Problem §5.1: 43: Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

Solution. This is a generalization of De Morgan's law: $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$. We will prove it via induction on the number of subsets, n. Let $P(n)$ denote the entire statement.

Base Case: The base case, $n = 1$, is simply the trivially true statement that $\overline{A_1} = \overline{A_1}$ because the union and intersection of a single set is just the set itself.

Inductive Step: Assume that for any collection of k subsets $A_1, \ldots, A_k$ of a universal set U, we have

$$\overline{\bigcup_{j=1}^{k} A_j} = \bigcap_{j=1}^{k} \overline{A_j}.$$

We wish to prove that for any collection of $k+1$ subsets, we have

$$\overline{\bigcup_{j=1}^{k+1} A_j} = \bigcap_{j=1}^{k+1} \overline{A_j}.$$

Beginning with the set on the left-hand side, we can observe that

$$\begin{aligned}\overline{\bigcup_{j=1}^{k+1} A_j} &= \overline{\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}} \\ &= \overline{\bigcup_{j=1}^{k} A_j} \cap \overline{A_{k+1}} \qquad \text{(by DeMorgan's law)} \\ &= \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cap \overline{A_{k+1}} \qquad \text{(by the IHOP)} \\ &= \bigcup_{j=1}^{k+1} \overline{A_j},\end{aligned}$$

Conclusion: Because we verified the base case $P(1)$ and showed that $P(k)$ implies $P(k+1)$ for all positive integers k, we conclude by the principle of mathematical induction that $P(n)$ holds for all positive integers n, as desired. □

Problem §6.2 - 44: There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

Solution. We use the pigeonhole principle. Consider the 100 numbers from 1000 to 1099, inclusive. 51 of these numbers are addresses and 49 are non-addresses. Let the ith "box" denote the numbers in between the $(i-1)$st and ith nonaddresses (where the 1st box starts at 1000). Then there are precisely 50 boxes, and since there are 51 addresses, at least one box must contain 2 or more addresses, which corresponds to 2 or more consecutive addresses.

(As an example of all of this, suppose the numbers are from 0-9 and the addresses are $0, 1, 3, 5, 7, 9$. The nonaddresses are $2, 4, 6, 8$, and so the boxes are $0-2, 2-4, 4-6, 6-8, 8-9$. Each box has one address except the first one has two.) □

Problem §6.3 - 33: Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Solution. We must have 3 women and 3 men, so the total number of committees is

$$\binom{15}{3} \cdot \binom{10}{3} = 455 \cdot 120 = 54600.$$

□

Problem §6.5 - 17: How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1s, and five 2s?

Solution. This is just the multinomial coefficient

$$\binom{10}{2,3,5} = \frac{10!}{2! \cdot 3! \cdot 5!} = 2520.$$

□

Problem §7.1 - 20: What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

Solution. We want to compute the probability of the event E that a hand contains the 10, jack, queen, king, and ace of one suit. Since the kinds of the cards are predetermined, we simply need to count the number of ways to choose the suit of the card. We can do so in $\binom{4}{1} = 4$ ways. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{4}{\binom{52}{5}} = \frac{1}{649,740}$$

□

Problem §7.1 - 30: What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of the six integers chosen at random from the integers between 1 and 40, inclusive?

Solution. The sample space S in this problem is the same as in Problem §7.1 - 27(a), i.e. the set of all possible choices of six integers (without repetition) from the set $\{1, 2, \dots, 40\}$.

In contrast, the event E is now the set of ways to choose six integers from $\{1, 2, \dots, 40\}$ such that exactly *five* of those integers are correct. Now, we can start by choosing five correct integers. There are $\binom{6}{5} = 6$ ways to do so. We then need to select the final incorrect integer, which we can do in $\binom{34}{1} = 34$ ways. Hence, $|E| = 6 \cdot 34$ and

$$p(E) = \frac{|E|}{|S|} = \frac{6 \cdot 4}{\binom{40}{6}} = \frac{17}{319,865} \approx 0.0053\%$$

□

Problem §7.2 - 38: A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

(a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

(b) Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

Solution. (a) The possible rolls are as follows: $16, 26, 36, 45, 56, 61, 62, 63, 64, 65, 66$. This is 11 possibilities, of which 2 have a sum of 7, so the probability of that sum is $\frac{2}{11}$.

(b) The possible rolls are as follows: $15, 25, 35, 45, 65, 51, 52, 53, 54, 56, 55$. This is 11 possibilities, of which 2 have a sum of 7, so the probability of that sum is again $\frac{2}{11}$.

(If we assume that at least one die came up five AND at least one die came up 6, then the sum is 11, so the probability of a sum of 7 is 0.)

□

Problem §8.1 - 28: Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4}+3f_{n-5}$ for $n = 5, 6, 7, \dots$, together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \dots$.

Solution. Using the Fibonacci recurrence $f_n = f_{n-1} + f_{n-2}$ repeatedly, if $n \geq 5$, then

$$f_n = f_{n-1} + f_{n-2} = 2f_{n-2} + f_{n-3} = 3f_{n-3} + 2f_{n-4} = 5f_{n-4} + 3f_{n-5},$$

as desired. Since this recursion start working for f_5, we need the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$.

We prove that f_{5n} is divisible by 5 by induction on n.

Base case: If $n = 0$ then $f_0 = 0$ is divisible by 5 (and if $n = 1$, then $f_5 = 5$ is too).

Inductive step: Assume that f_{5k} is divisible by 5, say that $f_{5k} = 5a$. Then

$$f_{5(k+1)} = f_{5k+5} = 5f_{5k+1} + 3f_{5k} = 5f_{5k+1} + 15a = 5(f_{5k+1} + 3a)$$

is divisible by 5, so this holds for all n by induction. □

Problem §8.5 - 18: How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?

Solution. See Theorem 1 in Section 8.5, and note that every intersection of some of the 10 sets appears, except for the empty intersection. Thus, there are a total of $2^{10} - 1 = 1023$ terms. □

Problem §9.1 - 55: Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n. (Here, R^n means $R \circ \cdots \circ R$, with n copies of R).

Solution. First we show that $R \circ R = R$ by showing that each is a subset of the other. If $(x, y) \in R$, then since R is reflexive, $(y, y) \in R$, so by composing these ordered pairs, we see that $(x, y) \in R \circ R$, and so $R \subseteq R \circ R$. On the other hand, if $(x, y) \in R \circ R$, then there exists some z such that $(x, z), (z, y) \in R$. But since R is transitive, $(x, y) \in R$, and so $R \circ R \subseteq R$.

Finally, we prove the result by induction.

Base case: $R^1 = R$ and by the previous paragraph, $R^2 = R \circ R = R$.

Inductive step: If $R^k = R$, then $R^{k+1} = R \circ R^k = R \circ R = R$.

Therefore, $R^n = R$ for all $n \geq 1$. □

Problem §9.3 - 23-26: List the ordered pairs in the relations represented by the directed graphs. (see Rosen)

Solution.

(23) $\{(a, b), (a, c), (b, c), (c, b)\}$

(24) $\{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)\}$

(25) $\{(a, c), (b, a), (c, d), (d, b)\}$

(26) $\{(a, a), (a, b), (b, a), (b, b), (c, a), (c, c), (c, d), (d, d)\}$

□

Problem §10.2.20: Draw the following graphs: $K_7, K_{1,8}, K_{4,4}, C_7, W_7, Q_4$

Solution. Available in person, by request. □

Problem §10.2.21-25: Determine whether the graph is bipartite

Solution. 21, 22, 24 are bipartite; the others are not. □

Problem §10.3.17: Draw an undirected graph represented by the given adjacency matrix (see Rosen)

Solution. Available in person, by request. □

Problem §10.4.2: Does each of these lists of vertices form a path in the following graph? (See Rosen!) Which paths are simple? Which are circuits? What are the lengths of those that are paths?

(a) a, b, e, c, b

(b) a, d, a, d, a

(c) a, d, b, e, a

(d) a, b, e, c, b, d, a

Solution. (a) Simple path, but not circuit, of length 4

(b) Non-simple circuit of length 4

(c) Not a path

(d) Not a path

□

Problem §10.5.46: Show that the Petersen graph G (see Rosen!) does not have a Hamilton circuit, but that the subgraph obtained by deleting a vertex v, and all edges incident with v, does have a Hamilton circuit.

Solution. *(Sketch)* Suppose that G has a Hamiltonian circuit. We will show that this is impossible. The Hamiltonian circuit must consist of 10 edges, and since G has 15 edges, there are 5 left over. Placing these 5 edges will always give a simple circuit of length 3 or 4, but G contains no such circuit.

If a is deleted, $b, c, d, e, j, h, f, i, g, b$ is a Hamiltonian circuit, and a similar argument follows for any other vertex. □

Problem §10.6.18: Is a shortest path between two vertices in a weighted graph unique if the weights of edges are distinct?

Solution. No. The simplest counterexample is C_3 with edge weights $1, 2, 3$. □

Problem §10.7.15: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, prove that $e \leq 2v - 4$.

Solution. This is very similar to the proof of Corollary 10.7.1. The big difference is that, because the graph has no circuits of length 3, the degree of each region is at least 4. Therefore, we have

$$2e = \sum_{\text{regions R}} \deg(R) \geq 4r,$$

and applying Euler's formula, $e \geq 2(e - v + 2)$, and solving for e, we have $e \leq 2v - 4$. □

Problem §11.1.2: Which of these graphs are trees?

Solution. a,b,d,f □

Problem §11.1.3 a-e: Answer these questions about the rooted tree illustrated (see Rosen!)

(a) Which vertex is the root?

(b) Which vertices are internal?

(c) Which vertices are leaves?

(d) Which vertices are children of j?

(e) Which vertex is the parent of h?

Solution. (a) a

(b) a, b, c, d, f, h, j, q

(c) All other vertices

(d) q, r

(e) c

□

Problem §11.1.14: Show that a simple graph is a tree if and only if it is connected but the deletion of any of its edges produces a graph that is not connected.

Solution. First, let G be a graph which is connected but the deletion of any of its edges produces a graph that is not connected. Then G has no simple circuits since if it did, we could delete an edge from a circuit and it would still be connected (just follow the circuit around the other way), so G is a tree.

Next, suppose G is a tree. Then by the properties we discussed in class, G is connected, and every edge is a cut edge, so the deletion of any of its edges produces a graph that is not connected. □

Problem §11.2.1: Build a binary search tree for the words banana, peach, apple, pear, coconut, mango, and papaya using alphabetical order.

Solution. Available in person, by request. □

Problem §11.2.33: Draw a game tree for nim if the starting position consists of two piles with two and three stones, respectively. When drawing the tree represent by the same vertex symmetric positions that result from the same move. Find the value of each vertex of the game tree. Who wins the game if both players follow an optimal strategy?

Solution. Available in person, by request. □