Note: the distribution of these problems may not match the distribution of exam topics. See also the lecture notes for some more of the solutions

Problem §2.1: Let $A = \{2, 6\}$ and $B = \{3, 1\}$.

(a) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$.

(b) Find $A \times B$.

(c) Is $A \times B = B \times A$? Why or why not?

Problem §2.1: Let $C = \{n \in \mathbb{N} : n < 6\}$ and $D = \{1, 3, 5\}$.

(a) Write C in set roster notation.

(b) Is $D \subseteq C$? Why or why not?

(c) Draw a Venn diagram representing sets C and D. (*Hint: This diagram should reflect the relationship that you determined in part (b).*)

Problem §2.2: Prove that $A \cup (A \cap B) = A$ by showing that each set is a subset of the other.

Problem §2.3: Consider the function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $z \mapsto |z| + 1$.

(a) Is f one-to-one? Why or why not?

(b) Is f onto? Why or why not?

Problem §3.1: 61: Write the deferred acceptance algorithm in pseudocode.

Solution. Let $A_1, \ldots, A_s$ denote the preference lists of the set of suitors and $B_1, \ldots, B_s$ denote the preference lists of the set of suitees. One possible way to write pseudocode for this algorithm is as follows:

```
procedure stable_matching(A_1, A_2, ..., A_s, B_1, B_2, ..., B_s : preference lists)
for i := 1 to s
    mark suitor i as rejected
for i := 1 to s
    set suitor i's rejection list to be empty
for j := 1 to s
    set suitee j's proposal list to be empty
while rejected suitors remain
    for i := 1 to s
        if suitor i is marked rejected then add i to the proposal
        list for the suitee j who ranks highest on suitor i's
        preference list but does not appear on their rejection list
        and mark suitor i as not rejected
    for j := 1 to s
        if suitee j's proposal list is nonempty then remove from j's
        proposal list all suitors i except the suitor i_0 who ranks
        highest on j's preference list, and for each such suitor i
        mark them as rejected and add j to their rejection list
    for j := 1 to s
        match j with the one suitor on j's proposal list
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□

Problem §3.2: Give a big-O estimate for $f(x) = (7n^n + n2^n + 3^n)(n! + 3^n)$. For the function $g(x)$ in your estimate $O(g(x))$, use a simple function g of the smallest order.

Problem §3.2: Use the definition of "$f(x)$ is $O(g(x))$" to show that $f(x) = x^4 + 7x^3 + 8$ is $O(x^4)$.

Problem §5.1: 10:

(a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

(b) Prove the formula you conjectured in part (a).

Solution. (a) To come up with a conjecture, we can try computing the value of this summation for small n:

$$\begin{aligned}
(n=1) &\quad \frac{1}{1 \cdot 2} = \frac{1}{2} \\
(n=2) &\quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
(n=3) &\quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}
\end{aligned}$$

Already, it seems like the appropriate formula might be

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(b) Now, let's prove this formula by induction.

Base Case: For $n = 1$, we've already checked that $1/(1 \cdot 2) = 1/2 = 1/(1+1)$.

Inductive Step: Assume that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

for $k \in \mathbb{Z}_{>0}$. We can then observe that

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(by the IHOP)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

as desired.

Conclusion: Because we verified that our formula held for $n = 1$ and that $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{Z}_{>0}$, we can conclude by the principle of mathematical induction that, for all $n \in \mathbb{Z}_{>0}$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

□

Problem §5.1: 18: Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

(a) What is the statement $P(2)$?

(b) Show that $P(2)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step.

(f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Solution. (a) The smallest integer greater than 1 is $n = 2$, so $P(2)$ is the base case for this statement. Plugging in $n = 2$, we see that $P(2)$ is the statement

$$2! < 2^2.$$

(b) Because $2! = 2$ and $2^2 = 4$, the statement that $2! < 2^2$ is just the true statement $2 < 4$.

(c) The inductive hypothesis is that $k! < k^k$ for some integer $k > 1$.

(d) For the inductive step, we need to prove that $P(k)$ implies $P(k+1)$ for each $k \geq 2$. That is, we want to prove that if we assume $k! < k^k$ is true, then $(k+1)! < (k+1)^{k+1}$ is as well.

(e) Observe that

$$\begin{aligned} (k+1)! &= (k+1)k! && \text{(by definition of the factorial function)} \\ &< (k+1)k^k && \text{(by the IHOP)} \\ &< (k+1)(k+1)^k \\ &= (k+1)^{k+1} \end{aligned}$$

(f) Because we verified that $2! < 2^2$ holds and that $k! < k^k$ implies $(k+1)! < (k+1)^{k+1}$ for all $k \geq 2$, we know by the principle of mathematical induction that $n! < n^n$ for all $n \geq 2$.

□

Problem §5.1: 22: For which nonnegative integers is $n^2 \leq n!$? Prove your answer.

Solution. To come up with a conjecture, we can start by testing some nonnegative integers. Observe that:

$$\begin{aligned}
0^2 &\leq 1 = 0! \\
1^2 &\leq 1 = 1! \\
2^2 &= 4 > 2! \\
3^2 &= 9 > 6 = 3! \\
4^2 &= 16 < 24 = 4! \\
5^2 &= 25 < 120 = 5!
\end{aligned}$$

From our computational exploration, it appears that the inequality holds for $n = 0, 1$ and all $n \geq 4$. We've already explicitly verified that the statement holds for $n = 0$ and $n = 1$. It remains for us to show that it holds for all $n \geq 4$. We'll do so by induction.

Base Case: Because we treated the $n = 0$ and $n = 1$ cases separately, the appropriate base case for our inductive argument is $n = 4$. Observe, as we saw above, that

$$4^2 = 16 < 24 = 4!,$$

so $P(4)$ holds.

Inductive Step: Assume that $k^2 < k!$ for some integer $k \geq 4$. Then observe that

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\leq k! + 2k + 1 \qquad \text{(by the IHOP)} \\
&\leq k! + 2k + k \qquad \text{(because } k \geq 4\text{)} \\
&= k! + 3k \\
&\leq k! + k \cdot k \qquad \text{(because } k \geq 4\text{)} \\
&\leq k! + k \cdot k! \\
&= (k+1) \cdot k! \\
&= (k+1)!
\end{aligned}$$

Conclusion: Because we verified the base case $n = 4$ and then showed that $k^2 < k!$ implies $(k+1)^2 < (k+1)!$ for all $k \geq 4$, we showed by mathematical induction that $n^2 < n!$ for all $n \geq 4$. We also independently verified that this inequality holds for $n = 0$ and $n = 1$. □

Problem §5.1: 36: Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

Solution. This problem is very similar to one of the examples that we did in class, so you can draw inspiration from that example when trying to find the algebraic trick needed in the inductive step.

We wish to show that 21 divides $4^{n+1} + 5^{2n-1}$ for all positive integers n.

Base Case: For $n = 1$, we can simply observe that

$$4^{1+1} + 5^{2 \cdot 1 - 1} = 4^2 + 5 = 16 + 5 = 21,$$

which is clearly divisible by 21.

Inductive Step: Now, assume that $21 \mid 4^{k+1} + 5^{2k-1}$ for some positive integer k. Then observe that

$$\begin{aligned}
4^{(k+1)+1} + 5^{2(k+1)-1} &= 4^{k+2} + 5^{2k+1} \\
&= 4 \cdot 4^{k+1} + 5^2 \cdot 5^{2k-1} \\
&= 4 \cdot 4^{k+1} + (4 + 21) \cdot 5^{2k-1} \\
&= 4\left(4^{k+1} + 5^{2k-1}\right) + 21 \cdot 5^{2k-1}
\end{aligned}$$

Now, consider the expression in the last line. We know by the inductive hypothesis that 21 divides $4^{k+1} + 5^{2k-1}$ and therefore divides the first term. The second term is clearly also divisible by 21, so we conclude that 21 divides the entire expression, as desired.

Conclusion: Because we verified the base case, when $n = 1$, and showed that 21 dividing $4^{k+1} + 5^{2k-1}$ implies that it also divides $4^{k+2} + 52k + 1$ for all positive integers k, we have shown by mathematical induction that 21 divides $4^{n+1} + 5^{2n-1}$ for all positive integers n. □

Problem §5.1: 38: Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j.$$

Solution. We will proceed by induction on the number of pairs of sets, n.

Base Case: Suppose that A_1 and B_1 are sets such that $A_1 \subset B_1$. Then because the union of one set is itself, this is trivially the statement that $\bigcup_{j=1}^{1} A_j = A_1 \subseteq B_1 = \bigcup_{j=1}^{1} B_j$.

Inductive Step: Assume that for any collection of sets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ such that $A_j \subseteq B_j$ for $j = 1, \ldots, k$, we have

$$\bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j.$$

Then consider some collection of sets $A_1, \ldots, A_{k+1}$ and $B_1, \ldots, B_{k+1}$ such that $A_j \subseteq B_j$ for $j = 1, \ldots, k+1$. We wish to show that

$$\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j.$$

To do so, we'll argue that any element of the first set, $\bigcup_{j=1}^{k+1} A_j$, must also be an element of the second set, $\bigcup_{j=1}^{k+1} B_j$. So, consider some

$$x \in \bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^{k} A_j \right) \cup A_{k+1}.$$

If $x \in \bigcup_{j=1}^{k} A_j$, then we know by the inductive hypothesis that $x \in \bigcup_{j=1}^{k} B_j$ and therefore $x \in \bigcup_{j=1}^{k+1} B_j$. If $x \in A_{k+1}$, then we know by the assumption $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$ and therefore $x \in \bigcup_{j=1}^{k+1} B_j$. Therefore, in either case $x \in \bigcup_{j=1}^{k+1} B_j$ and so

$$\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j.$$

Conclusion: Because we verified the statement for the base case $n = 1$ and showed that the statement holding for $n = k$ implies it also holds for $n = k + 1$ for all positive integers k, we have therefore shown by induction that the statement holds for all n, as desired. □

Problem §5.1: 57: Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer.

Solution. We wish to prove that the derivative of $f(x) = x^n$ is equal to nx^{n-1} whenever n is a positive integer.

Base Cases: Using the limit definition of a derivative, we can check the base case $n = 1$ by computing

$$\frac{d}{dx}\left(x^1\right) = \lim_{h\to 0} \frac{(x+h)^1 - x^1}{h} = \lim_{h\to 0} \frac{h}{h} = 1 = 1 \cdot x^0.$$

Inductive Step: Assume that for some positive integer $k > 1$, we have $\frac{d}{dx}(x^k) = kx^{k-1}$. Then observe that

$$\begin{aligned}
\frac{d}{dx}\left(x^{k+1}\right) &= \frac{d}{dx}\left(x^k \cdot x\right) \\
&= x \cdot \frac{d}{dx}\left(x^k\right) + x^k \cdot \frac{d}{dx}(x) \qquad \text{(by the product rule)} \\
&= x \cdot kx^{k-1} + x^k \cdot 1 \qquad \text{(by the IHOP and base case)} \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

Conclusion: Because we verified the base case $\frac{d}{dx}(x) = 1 \cdot x^0$ and showed that assuming $\frac{d}{dx}(x^k) = k \cdot x^{k-1}$ implies $\frac{d}{dx}(x^{k+1}) = (k+1) \cdot x^k$ for all positive integers k, we have therefore shown by the principle of mathematical induction that $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ for all positive integers n, as desired. ☐

Problem §5.2: 31: Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

Solution. We can essentially use the same tactic as we used in class to prove that mathematical induction is a valid method of proof. We'll give a proof by contradiction.

Suppose that we have some proposition $P(n)$ that we proved is true for all n using strong induction. We want to show that $P(n)$ is actually true for all n, so our proof via strong induction was valid. Let

$$S := \{n \;:\; P(n) \text{ is false}\}.$$

If strong induction is valid, then $S = \emptyset$. By way of contradiction, assume that S is nonempty. Then by the well-ordering property, S has a least element, m. Because m is the *smallest* element of S, all of the statements $P(b), \ldots, P(m-1)$ must be true. But when we gave a proof of $P(n)$ via strong induction, we showed that if $P(b), \ldots, P(m-1)$ are true, then $P(m)$ is as well. This contradicts our assumption that $m \in S$. Hence, we are forced to conclude that $S = \emptyset$, that $P(n)$ is true for all n, and that strong induction is therefore a valid method of proof. ☐

Problem §6.1: 28: How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

Problem §6.1: 37: How many functions are there from the set $\{1, 2, ..., n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) That are one-to-one?

(b) That assign 0 to both 1 and n?

(c) That assign 1 to exactly one of the positive integers less than n?

Problem §6.2: 31: Show that there are at least six people in California (population: 37 million) with the same three initials who were born on the same day of the year (but not necessarily the same year). Assume that everyone has three initials.

Solution. Here, the "pigeons" are the people living in California and the "pigeonholes" are all the possible pairs of initials and days of the year. In order to apply the pigeonhole principle, we need to count the number of pigeonholes. Because there are 26 ways to choose each of the 3 initials and 366 ways to choose the birthday, the product rule tells us that there are $26^3 \cdot 366 = 6,432,816$ possible combinations. The generalized pigeonhole principle then tells us that there must be at least

$$\left\lceil \frac{37,000,000}{6,432,816} \right\rceil = 6$$

people with the same combination. □

Problem §6.2: 40: Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution. Suppose there are $n \geq 2$ people at the party. Assume that "knowing" is symmetric, so either two people mutually know each other or they don't. Let K be a function that maps each person x at the party to the number of people that they know, $K(x)$. We would like to apply the pigeonhole principle, where the "pigeons" are the people at the party and the "pigeonholes" are the number of people that they know at the party.

On the face of it, the possible values for $K(x)$ are $0, 1, \ldots, n-1$ (i.e., ranging from knowing nobody at the party to knowing everyone). However, it's not possible for both 0 and $n-1$ to appear in the range of K, because if one person at the party knows nobody then it's not possible for another person to know everyone. Hence, the range of K contains $n-1$ elements. Because there are n people at the party, we have more "pigeons" than "pigeonholes" and therefore the pigeonhole principle guarantees that there are at least two people at the party who know the same number of other people there. □