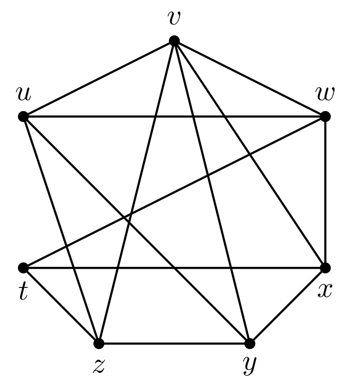
Solutions to Math 213-X1 Midterm Exam 3 — Nov. 20, 2024

1. (16 points) Consider the following graph G.



(a) (7 points) Determine whether G has an Eulerian circuit, an Eulerian path, or neither. *(You are not required to find the circuit/path if it exists, although you may)*

The degree sequence of G is $5, 4, 4, 4, 4, 4, 3$, so G has exactly two vertices of odd degree, and therefore has an Eulerian path but not an Eulerian circuit.

(b) (7 points) Determine whether G has a Hamiltonian circuit, a Hamiltonian path, or neither. *(You are not required to find the circuit/path if it exists, although you may)*

G has the following Hamiltonian circuit: t, x, y, z, u, v, w, t. This circuit is Hamiltonian since it uses each vertex exactly once (except for t, which is the first and last vertex).

(c) (2 points) What are the cut vertices of G? *(No work needed for this part)*

There are none!

2. (18 points) Let $A = \{a, b, c\}$. Find a relation on A (which may be expressed as a set of ordered-pairs, as a matrix, or as a digraph) that has the following properties.

(No work necessary for this problem! Only your answer will be graded, based on which of the properties it's correct on)

(a) (6 points) Symmetric and transitive, but not reflexive

Any relation which is a proper subset of $\{(a, a), (b, b), (c, c)\}$ satisfies these conditions. Adding some of the other ordered pairs sometimes works, but risks violating symmetry or transitivity.

(b) (6 points) Reflexive and symmetric, but not transitive

$\{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$ or similar.

(c) (6 points) Asymmetric and transitive, and containing at least three ordered pairs

$\{(a, b), (b, c), (a, c)\}$, or similar.

3. (12 points) Let A be the set of all binary strings of length 4. Let R be the equivalence relation on A given by $(x, y) \in R$ if and only if x and y agree in the first digit and last digit.

(No work necessary for this problem! Only your answer will be graded.)

(a) (4 points) What is the equivalence class of the string 0101?

$\{0001, 0011, 0101, 0111\}$

(b) (4 points) How many equivalence classes are there in total?

4

(c) (4 points) True or false: Let x, y be distinct (i.e. different) elements of A. If x and y agree in the two middle digits, they must be in different equivalence classes.

True

4. (12 points) Let $A = \{a, b, c\}$, and consider the relation $R = \{(a, b), (a, c), (b, a), (b, b)\}$. Give the matrix and digraph corresponding to R, and the matrix and digraph corresponding to the composition $R \circ R$.

First we do the matrices. From the ordered pairs,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

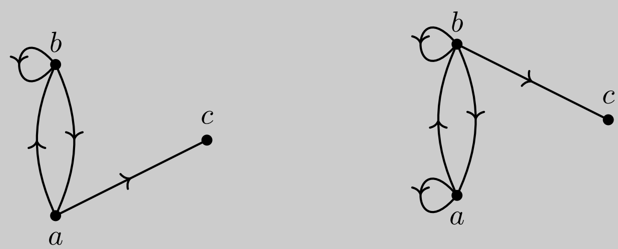
Then,

$$M_R \cdot M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

and using the boolean product, we get

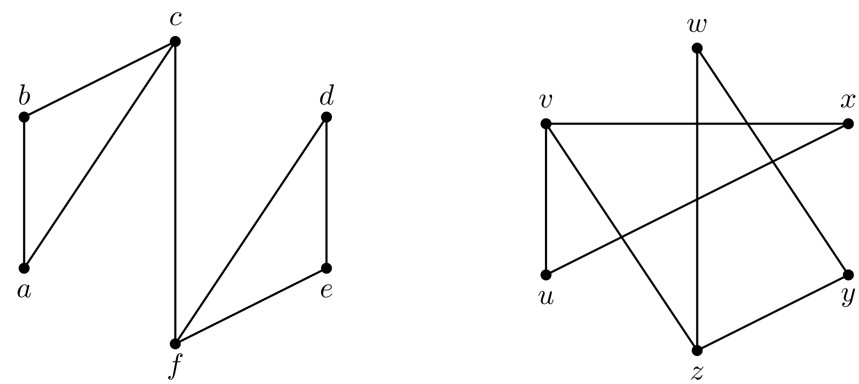
$$M_{R \circ R} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Now we know R and (the matrix for) $R \circ R$, so we can write down their corresponding digraphs. The one for R is on the left; the one for $R \circ R$ is on the right:



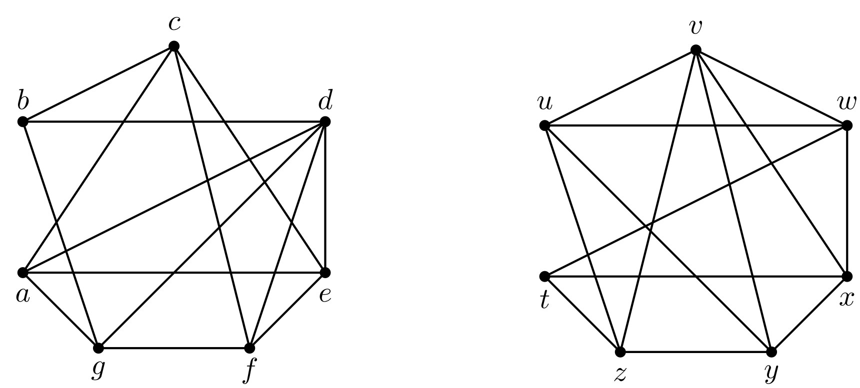
5. (20 points) Determine if the given pair of graphs is isomorphic. Either exhibit an isomorphism or provide a rigorous argument that none exists.

(a) (10 points)



These graphs are isomorphic. Let p be the function $p(a) = x, p(b) = u, p(c) = v, p(d) = z, p(e) = y, p(f) = w$. Then p is a bijection from the vertex set of the left graph to the vertex set of the right graph. Furthermore, two vertices m and n are adjacent in the left graph if and only if $p(m)$ and $p(n)$ are adjacent in the right graph. Therefore, p is a graph isomorphism, and so the graphs are isomorphic.

(b) (10 points)



These graphs are not isomorphic, despite having the same degree sequences. To see that they aren't isomorphic, note that the left graph has exactly one vertex of degree 3 (b), and exactly one vertex of degree 5 (d), and these vertices are adjacent. On the other hand, the right graph has exactly one vertex of degree 3 (t), and exactly one vertex of degree 5 (v), but these vertices are *not* adjacent. Therefore, the graphs are not isomorphic since any isomorphism p must preserve degrees, and so we would have $p(b) = t$ and $p(d) = v$, but b and d are adjacent while t and v aren't.