Solutions to Math 213-X1 Midterm Exam 2 — Oct. 25, 2024

1. (20 points) Find all solutions to the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3^n$.

This is a linear inhomogeneous recurrence relations, so all solutions are of the form $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(p)}$ is any particular solution and $a_n^{(h)}$ is a solution to the homogeneous recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}.$$

The characteristic equation is $r^2 - 4r + 4 = (r-2)^2$, so the only root is $r = 2$, with multiplicity 2. Therefore, (by Theorem 8.2.4) the general solution to the homogeneous equation is $a_n^{(h)} = (\alpha + \beta n)2^n$, where α and β are arbitrary.

Next, for the particular solution. Since the inhomogeneous part is $F(n) = 3^n$ and 3 is not a root of the characteristic equation, we know (Theorem 8.2.6) that there is a particular solution of the form $a_n^{(p)} = p3^n$, for some (but not all!) values of p.

Finally, we plug this particular solution into the recurrence relation to obtain

$$p3^n = 4p3^{n-1} - 4p3^{n-2} + 3^n,$$

so factoring out 3^{n-2}, $9p = 12p - 4p + 9$, and solving for p we get $p = 9$.

Therefore, the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha + \beta n)2^n + 9 \cdot 3^n,$$

for arbitrary α and β.

2. (10 points) Suppose that the word "bitcoin" appears in 200 out of 1,000 spam email messages and in 1 out of 1,000 legitimate email messages. If a randomly chosen message is just as likely to be spam as to be legitimate, what is the probability that a given message containing the word "bitcoin" is spam?

We use Bayes' Theorem. Given an email, let E be the event that it is spam, and let F be the event that it contains the word "bitcoin". We want to find $p(E|F)$

From the problem statement, $p(E) = p(\overline{E}) = 0.5$, $p(F|E) = 200/1000 = 0.2$, and $p(F|\overline{E}) = 1/1000 = 0.01$.

Applying Bayes Theorem, we have

$$p(E|F) = \frac{p(F|E)p(E)}{p(F)} = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\overline{E})p(\overline{E})} = \frac{0.2 \cdot 0.5}{0.2 \cdot 0.5 + 0.001 \cdot 0.5} = 0.995.$$

(Since no calculators are allowed, $\frac{0.2 \cdot 0.5}{0.2 \cdot 0.5 + 0.001 \cdot 0.5}$ is an acceptable answer)

3. (10 points) Let A, B, and C be sets with $|A| = 9$, $|B| = 8$, $|C| = 10$, $|A \cap B| = 6$, $|A \cap C| = 3$ $|B \cap C| = 4$, and $|A \cap B \cap C| = 2$. Find $|A \cup B \cup C|$.

By inclusion-exclusion,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 9 + 8 + 10 - 6 - 3 - 4 + 2 \\ &= 16 \end{aligned}$$

4. (10 points) Recall that a standard deck contains a total of 52 cards, 4 each of the kinds: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. The kinds Jack, Queen, and King are called *face cards*.

   What is the probability that a random five-card hand contains exactly 2 face cards?

   (*For this problem, you may leave your answer in terms of binomial coefficients.*)

   There are a total of $\binom{52}{5}$ poker hands. Each deck contains $3 \cdot 4 = 12$ face cards, and 40 non-face cards, so to choose a hand with exactly 2 face cards, we choose 2 of the 12 face cards and 3 of the 40 non-face cards; thus there are $\binom{12}{2}\binom{40}{3}$ possible such hands.

   Therefore, the probability that a given hand has exactly two face cards is

$$\frac{\binom{12}{2}\binom{40}{3}}{\binom{52}{5}}.$$

5. (20 points) Answer the following questions.

   (No work necessary for this problem! Only your answer will be graded.)

   (For this problem, you may leave your answer in terms of binomial coefficients.)

   (a) (5 points) How many ways are there to buy a total of 10 cookies when there are 4 different flavors?

   Sticks-and-stones: $\binom{10+4-1}{10} = \binom{13}{10}$.

   (b) (5 points) How many ways are there to pack 4 identical copies of a book into any number of indistinguishable boxes?

   This is just the number of ways to write 4 as a sum of positive integers in decreasing order: 4; $3+1$; $2+2$; $2+1+1$; $1+1+1+1$. Total: 5

   (c) (5 points) How many ways are there to pack 16 distinguishable objects into 20 distinguishable boxes?

   Repeated product rule gives 20^{16}

   (d) (5 points) How many permutations of the letters $ABCDEF$ contain either the string AB or the string BA? (Without any other letters in between)

   Since we can't have both AB and BA, no need to do inclusion-exclusion. For strings containing AB, just treat that as a chunk. Then we have 5 chunks, so there are 5! such permutations. Same for BA, so we have a total of $2 \cdot 5! = 240$ valid permutations.

6. (15 points) Let n and k be positive integers. **Using a combinatorial argument**, prove the following identity:

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k}\binom{n-j}{k}$$

   We count the number of binary strings of length $n+1$ with $2k+1$ 1's and $n-2k$ 0's. We do this in two ways. On one hand, choosing the positions of the 1's clearly gives $\binom{n+1}{2k+1}$.

   On the other hand, let $j+1$ be the position of the *middle* 1, i.e. the $(k+1)$st 1 reading from left to right. There are k 1's both before and after position $j+1$, so we must have $k < j+1 \le n+1-k$, so $k \leq j \leq n-k$. Then we can (independently) choose the k 1's before position $j+1$, and the k 1's after position $j+1$. The former choice has $\binom{j}{k}$ possibilities, while the latter has $\binom{n-j}{k}$ possibilities. Since we are counting the same set in multple ways, we have

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k}\binom{n-j}{k},$$

as desired.