Solutions to Math 213-X1 Midterm Exam 1 — Sept. 25, 2024

1. (20 points) (a) (15 points) Prove that for any sets A and B that

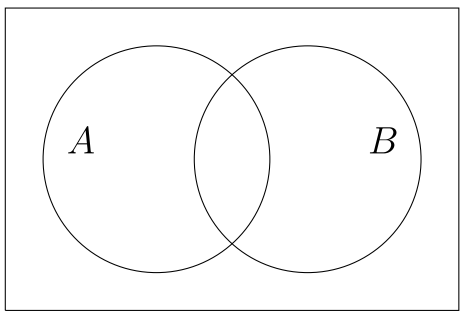
$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

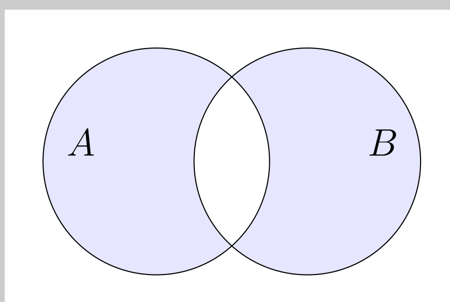
($A - B$ can equivalently be written $A \setminus B$)

We show that $(A-B)\cup(B-A) \subseteq (A\cup B)-(A\cap B)$ and that $(A\cup B)-(A\cap B) \subseteq (A-B)\cup(B-A)$. First, let $x \in (A - B) \cup (B - A)$. Then either $x \in A - B$ or $x \in B - A$. In the first case, $x \in A$ and $x \notin B$, so $x \in A \cup B$ and $x \notin A \cap B$, so $x \in (A \cup B) - (A \cap B)$. In the second case, $x \in B$ and $x \notin A$, so $x \in A \cup B$ and $x \notin A \cap B$, so $x \in (A \cup B) - (A \cap B)$. Therefore, $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

Now let $x \in (A \cup B) - (A \cap B)$. Since $x \in A \cup B$, either $x \in A$ or $x \in B$. In the first case, since $x \notin A \cap B$, we must have $x \notin B$, so $x \in A - B$, so $x \in (A - B) \cup (B - A)$. In the second case, since $x \notin A \cap B$, we must have $x \notin A$, so $x \in B - A$, so $x \in (A - B) \cup (B - A)$.

(b) (5 points) Shade the Venn diagram below to represent the set above. *(Be very clear about which region(s) are shaded and which aren't)*





2. (15 points) Use strong induction to prove the following statement: Any amount of postage greater than or equal to 8 cents can be formed using 3-cent and 5-cent stamps. Clearly indicate your base case(s), inductive hypothesis, and inductive step.

Let $P(n)$ be the statement that n cents can be formed with 3-cent and 5-cent stamps. We prove that $P(n)$ is true for all $n \geq 8$ by strong induction.

For the base cases,

$$8 = 3 + 5, \qquad 9 = 3 + 3 + 3, \qquad 10 = 5 + 5$$

so $P(8), P(9)$, and $P(10)$ are all true.

Now for the inductive step, let $k \geq 10$, and assume (inductive hypothesis) that $P(n)$ is true for all $8 \leq n \leq k$. Then in particular, $P(k+1-3)$ is true since $k+1-3 < k$ and $k+1-3 \geq 10+1-3 = 8$. Therefore, there exists a way to form $k+1-3$ cents via 3-cent and 5-cent stamps. Take that set of stamps and add a 3-cent stamp; the resulting stamps have a value of $k+1$ cents. Thus, $P(k+1)$ is true, and therefore, $P(n)$ is true for all $n \geq 8$ by strong induction.

3. (20 points) Answer the following questions. *(No work necessary for this problem!)*

   (a) (4 points) Let $A = \{3, 4\}$. What is $\mathcal{P}(A)$?

   $\{\emptyset, \{3\}, \{4\}, \{3, 4\}\}$

   (b) (4 points) List three of the seven properties that algorithms must have (according to Rosen).

   Input, output, finiteness, correctness, definiteness, effectiveness, generality

   (c) (4 points) Find the smallest integer n such that $f(x) = (3x^2 + 10)(7x \log x + x + 2)$ is $O(x^n)$.

   $n = 4$

   (d) (4 points) How many one-to-one functions are there from A to B if $|A| = |B| = 4$?

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   (e) (4 points) Given that a deck of cards has 4 suits, what is the smallest number of cards we need to draw to guarantee that at least 5 of them are the same suit?

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4. (10 points) Prove that $f(x) = 3x^4 + 7x^2 + 9$ is $O(x^4)$.

Let $k = 1, C = 20$. Then for $x > k$, $x^4 > x^2$ and $x^4 > 1$, so when $x > k$,

$$\begin{aligned} |f(x)| &= 3x^4 + 7x^2 + 9 \\ &\leq 3x^4 + 7x^4 + 9x^4 \\ &= 19x^4 \\ &< 20x^4 \\ &= C|x^4|, \end{aligned}$$

so $f(x)$ is $O(x^4)$.

5. (20 points) Determine the following quantities, with justification. *(Arithmetical expressions such as $9^2 + 2 \cdot 8^2$ that can be directly plugged into a calculator need not be simplified).*

   (a) (5 points) How many 6-digit codes are there consisting only of the digits $1, 2, 3, 4$ and 5? (e.g. 111111, 132453)

   There are 5 choices for each of the 6 digits, so by the product rule, there are 5^6 codes in total.

   (b) (15 points) How many of these codes either start with three consecutive 5's or end with three consecutive NON-5's?

   The codes which start with three consecutive 5's are of the form $555def$, where $d, e, f \in \{1, 2, 3, 4, 5\}$, so by the product rule, there are 5^3 such codes.
   The codes which end with three consecutive non-5's are of the form $abcdef$ where $a, b, c \in \{1, 2, 3, 4, 5\}$ and $d, e, f \in \{1, 2, 3, 4\}$. Therefore, there are 5 choices for each of the first three entries and 4 choices for each of the last three entries. By the product rule, there are $5^3 \cdot 4^3$ such codes.
   Finally, the codes which have BOTH of the above forms are of the form $555def$ where $d, e, f \in \{1, 2, 3, 4\}$, so by the product rule, there are 4^3 such codes.
   Therefore, by the subtraction principle, there are

   $$5^3 + 5^3 \cdot 4^3 - 4^3 = 8061$$

   valid codes. (No need to simplify the expression on the left).