Announcements:

Quiz 3 this Wed.

Midterm 1 Wed 9/25 in-class (50 minutes)

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§ 5.2: Strong Induction and Well-Ordering

Recall:

Principle of Mathematical Induction:

$P(n)$ is true for all n if and only if

- $P(1)$ is true (base case)
- If we assume $P(k)$ is true then $P(k+1)$ is true (induction step)

Strong induction:

$P(n)$ is true for all n if and only if

- $P(1)$ is true (base case)
- If we assume $P(1), P(2), \ldots, P(k)$ are all true, then $P(k+1)$ is true (induction step)

Only difference: now we get to assume $P(1), \ldots, P(k)$ in the induction step instead of just $P(k)$

No downside to using strong induction

Sometimes just assuming $P(k)$ is enough

Other times, it helps to assume $P(1), \ldots, P(k)$

Ex 2: Show that if $n \in \mathbb{Z}$, $n \geq 2$, then n can be written as a product of one or more primes

Pf: We prove this by induction. Let $P(n)$ be the statement "n can be written as a product of primes"

Base case: 2 is prime, so $P(2)$ is true

Inductive step: Assume that $P(1), \ldots, P(k)$ are true.

If $k+1$ is prime, then $P(k+1)$ is true. Otherwise, $k+1 = ab$ for integers $2 \leq a, b < k+1$. By the inductive hypothesis, a and b can be written as products of primes:

$$a = p_1 p_2 \cdots p_n, \quad b = q_1 q_2 \cdots q_m.$$

Then

$$k+1 = p_1 \cdots p_n q_1 \cdots q_m$$

can also be written as a product of primes, so $P(k+1)$ is true, and $P(n)$ is true for all $n \geq 2$ by induction

$\square$

Ex 4: Prove that every amount of postage of 12 cents or more can be formed using just 4 cent and 5 cent stamps

Pf: We prove this via strong induction. Let $P(n)$ be the statement "n cents can be formed using just 4-cent & 5-cent stamps"

Base cases: $12 = 4+4+4$

$13 = 4+4+5$

$14 = 4+5+5$

$15 = 5+5+5$

So $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are all true.

Inductive step: Let $k \geq 15$, and assume that $P(12), \ldots, P(k)$ are all true. We want to show that $P(k+1)$ is true. Since $k \geq 15$, $k+1 \geq 16$, so $k+1-4 \geq 12$, so $P(k+1-4)$ is true. Therefore, we can make $k+1-4$ cents using 4-cent and 5-cent stamps, so adding a 4-cent stamp gives $k+1$ cents. Thus, $P(k+1)$ is true, so $P(n)$ is true for all $n \geq 12$ by strong induction □

Ex 3: Consider the following game: Two piles of n matches

The players take turns removing ≥ 1 matches from one of the piles. The players who takes the last match wins. Show that Player 2 can always guarantee a win.

Class activity: play this game, and try to figure out a strategy.

Pf: We use strong induction. Let $P(n)$ be "Player 2 can win whenever there are initially n matches in each pile"

Base case: If $n=1$, Player 1 must remove the 1 match from one of the piles. Player 2 takes the match from the other pile and wins.

Inductive step: Suppose $k \geq 1$ and $P(1), \dots, P(k)$ are true. For $k+1$ matches per pile, suppose Player 1 takes r matches from the first pile. Then Player 2 can take r matches from the other pile. If $r = k+1$, Player 2 wins. If $1 \leq r < k+1$, then each pile has $k+1-r$

matches remaining, and it is Player 1's turn again. Since $1 \le k+1-r \le k$, $P(k+1-r)$ is true, so Player 2 can now guarantee a win. Thus, $P(k+1)$ is true, so by strong induction, $P(n)$ is true for all n. $\square$

If time:

Well-Ordering Property: Every nonempty subset A of $\mathbb{N}$ has a smallest element.

Note: not true for subsets of $\mathbb{Z}$ e.g. $\{\dots, -8, -6, -4, \dots\}$
or for subsets of $\mathbb{R}_+$ e.g. $\{x \in \mathbb{R} \mid 0 < x < 1\}$

Ex 5: Use the well-ordering property to prove the division algorithm:

If $a \in \mathbb{Z}$, $d \in \mathbb{Z}_+$, then there are (unique) integers q, r with $0 \le r < d$ such that
$$a = qd + r$$

e.g. $a = 31$, $d = 3$

$$31 = \underbrace{10}_{q} \cdot 3 + \underbrace{1}_{r} \qquad \text{(division w/ remainder)}$$

Pf: Let S be the set of nonnegative integers of the form $a-dq$, where q is an integer.

S is nonempty since taking $q \ll 0$ makes $a-dq \geq 0$.

By the well-ordering property, S has a smallest element $r = a - dq_0$. $r \in \mathbb{N}$ since $r \in S$. In addition, if $r \geq d$, then $r_1 := a - d(q_0+1) \in S$ and $r_1 < r$, but this can't happen since r is the smallest elt. $\square$