# §5.1: Mathematical Induction

(Monday, we'll discuss a variant called <u>strong induction</u>)

Ex 1: Show that if $n$ is a positive integer, then

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Let's check a couple of cases:

$n=1$: $1 = \frac{1\cdot 2}{2}$ ✓

$n=2$: $1+2 = 3 = \frac{2\cdot 3}{2}$ ✓

$n=3$: $1+2+3 = 6 = \frac{3\cdot 4}{2}$ ✓

Seems like it probably works

In this case, there's a trick:

$$\left.\begin{array}{r} 1+2+\cdots+n \\ +\quad n+n-1+\cdots+1 \end{array}\right\} 2\cdot LHS$$

$$\overline{n+1+n+1+\cdots+n+1} = n(n+1)$$

But in general, we want a better tool

$n=3$ : LHS $= 1+2+3$ RHS $= \frac{3\cdot 4}{2}$

$\Downarrow$

$n=4$ : LHS $= 1+2+3+4$ RHS $= \frac{4\cdot 5}{2}$

bigger by 4

bigger by $\frac{4\cdot 5}{2} - \frac{3\cdot 4}{2} = \frac{4}{2}(5-3) = 2\cdot 2 = 4$

What about for general $n$?

Assume that

$$1+2+\cdots+n = \frac{n(n+1)}{2} \quad (*)$$

WTS:

$$1+2+\cdots+n+n+1 = \frac{(n+1)(n+2)}{2}$$

$$\begin{aligned} 1+2+\cdots+n+n+1 &= \frac{n(n+1)}{2} + n+1 \qquad \text{(by } *\text{)} \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \quad \checkmark \end{aligned}$$

So if the equation holds for $n=1$ AND whenever it holds for $n$ it holds for $n+1$, it <u>must</u> hold for all $n$.

Let $P(n)$ be a statement (true or false) depending on the positive integer $n$.

Want to show that $P(n)$ is true for all $n$

Principle of Mathematical Induction:

$P(n)$ is true for all $n$ if and only if

- $P(1)$ is true (base case)
- If we assume $P(k)$ is true (for arbitrary $k$), then $P(k+1)$ is true (induction step)

Ex 1 (cont).

Pf: Let $P(n)$ be the statement

$$1+\cdots+n=\frac{n(n+1)}{2}$$

We prove $P(n)$ is true for all $n$ by induction on $n$.

Base case: When $n=1$,

$$1=\frac{1\cdot 2}{2}, \text{ so } P(1) \text{ is true.}$$

Inductive step: Assume that $P(k)$ is true. Then,

$$1+\cdots+k+(k+1)=\frac{k(k+1)}{2}+k+1 \quad (\text{by } P(k))$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2},$$

so $P(k+1)$ is true. Therefore, $P(n)$ is true for all $n$ by induction. $\square$

Def: The statement $P(k)$ in the inductive step is called the <u>inductive hypothesis</u> (since we assume it's true)

Remark: The textbook has more on the history/philosophy of induction

Ex 2: Find and prove a formula for the sum of the first $n$ odd integers $1+3+\dots+(2n-1)$

$n=1$: $1$

$n=2$: $1+3=4$

$n=3$: $1+3+5=9$

$n=4$: $1+3+5+7=16$

Let $P(n)$ be the statement:

$$1 + 3 + \cdots + (2n-1) = n^2$$

We prove $P(n)$ for all $n$ by induction

Base case: $1 = 1^2$, so $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true. Then,

$$\begin{aligned} 1 + 3 + \cdots + (2k-1) + (2k+1) &= k^2 + (2k+1) \quad \text{(by the inductive hypothesis } P(k)\text{)} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

so $P(k+1)$ is true, and so $P(n)$ is true for all $n$ by induction.

$\square$

Ex 6: Prove that $2^n$ is $O(n!)$.

| $n$ | $2^n$ | $n!$ |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 4 | 2 |
| 3 | 8 | 6 |
| 4 | 16 | 24 |
| 5 | 32 | 120 |

Pf: Let $k=4$, $C=1$. We prove that $2^n$ is $O(n!)$ by showing that $|2^n| < C|n!|$ for all $n > k$.

Let $P(n)$ be the statement $2^n < n!$. We want to prove that $P(n)$ is true for all $n \geq 5$. We prove this by induction.

Base case: $n = 5$ (note: modified starting point!)

When $n=5$, $2^n = 32 < 120 = n!$, so $P(5)$ is true.

Inductive step: Suppose that $P(a)$ is true, and $a \geq 5$.

We want to show $P(a+1)$ is true. We have,

$$2^{a+1} = 2 \cdot 2^a < 2 \cdot a! < (a+1)a! = (a+1)!,$$

so $P(a+1)$ is true. Therefore, $P(n)$ is true for all $n \geq 5$ by induction, so $2^n$ is $O(n!)$. $\square$

Ex 8: Prove that $\underbrace{n^3 - n \text{ is divisible by } 3}_{3 \mid n^3 - n}$ for all positive integers $n$.

Pf: Let $P(n)$ be the statement $3 \mid n^3 - n$. We prove that $P(n)$ is true for all $n$ by induction.

Base case: If $n = 1$, $n^3 - n = 1^3 - 1 = 0 = 0 \cdot 3$, so

$P(1)$ is true.

Inductive step: Assume $P(k)$ is true. Then, $3 \mid k^3 - k$, so let $k^3 - k = 3m$, where $m$ is an integer.

Then,

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3m + 3(k^2 + k) \\ &= 3(m + k^2 + k)\end{aligned}$$

So $(k+1)^3 - (k+1)$ is divisible by 3, and $P(k+1)$ is true.

Thus, $P(n)$ is true for all $n$ by induction. $\square$