Announcements

HW3 posted (due Sunday 11:59 pm)

Quiz 1 today!

§3.2: The growth of functions

Def (big-O notation): Let f, g be functions from $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$.

Idea: $f(x)$ is $O(g(x))$ if "eventually", $g(x)$ is bigger. If they are "the same" up to a constant, then f is $O(g)$ and g is $O(f)$.

"Omega"

Def: If f is $O(g)$, then g is $\Omega(f)$

"Theta"

If f is $O(g)$ and g is $O(f)$, then f is $\Theta(g)$

Think: f is $O(g) \leftrightarrow f \leq g$
f is $\Omega(g) \leftrightarrow f \geq g$
f is $\Theta(g) \leftrightarrow f = g$

this is <u>not</u> literally true, just a good way to think of it

Examples:

a) x^2 is $O(x^3)$, so x^3 is $\Omega(x^2)$

b) $3x^2+17x+6$ is $O(x^{2.1})$

c) $3x^2+17x+6$ is $O(x^2)$ and x^2 is $O(3x^2+17x+6)$

so $3x^2+17x+6$ is $\Theta(x^2)$

d) x^a is $O(e^x)$ for any a

e) $n!$ is $\Omega(e^n)$

n	e^n	$n!$
1	e	1
2	e^2	$1\cdot 2$
3	e^3	$1\cdot 2\cdot 3$
4	e^4	$1\cdot 2\cdot 3\cdot 4$
5	e^5	$1\cdot 2\cdot 3\cdot 4\cdot 5$

f) x^a is $\Omega(1)$ if $a \geq 0$

x^a is $O(1)$ if $a \leq 0$

g) $\log x$ is $O(x)$

Simple tricks:

1) Larger powers grow faster
2) Ignore constant factors
3) Only worry about the fastest-growing term

Now for some proofs:

Ex 1: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$

Pf: We need to find $C, k \in \mathbb{R}$ s.t.

$$|f(x)| \leq C|x^2| \text{ whenever } x > k$$

Let $k = 10$, $C = 5$. Then if $x > k$,

$$f(x) = x^2 + 2x + 1$$

$$< x^2 + 2x^2 + x^2 \qquad (\text{since } x \geq k \geq 1)$$

$$= 4x^2$$

$$< 5x^2$$

$$= C|x^2|.$$

Therefore, $f(x)$ is $O(x^2)$. $\square$

Note: we chose C and k much bigger than needed. $k = 1, C = 4$ or $k = 3, C = 2$ would have worked.

Ex 9: Show that $f(x) = (x+1)\log(x^2+1)$ is $\Theta(x \log x)$.

Pf: We show that a) $f(x)$ is $O(x\log x)$ and b) $x \log x$ is $O(f(x))$.

b) Notice that $\log x$ is an increasing function. Let $k = 1$, $C = 1$. Then if $x > k$,

$$\begin{aligned} x\log x &\leq x \log(x^2) && (\text{since } x^2 \geq x) \\ &\leq x\log(x^2+1) && (\text{since } x^2+1 > x^2) \\ &\leq (x+1)\log(x^2+1) && (\text{since } x+1 > x) \\ &= C|f(x)| \end{aligned}$$

a) Let $k =$, $C =$. Then if $x > k$,

$$\begin{aligned} (x+1)\log(x^2+1) &\leq (x+1)\log(2x^2) && (\text{since } 1 \leq x^2) \\ &= (x+1)(\log 2 + \log x + \log x) && (\text{by log rules}) \\ &\leq (x+1)\, 3\log x && (\text{since } x > 2, \text{ so } \log x > \log 2) \\ &< 2x \cdot 3 \log x && (\text{since } x > 1) \\ &= 6x\log x \end{aligned}$$

$\square$

Ex 11: Let $f(n) = 1 + 2 + \cdots + n$. Show that f is $\Theta(n^2)$.

Pf: We show that a) f is $O(n^2)$ and b) f is $\Omega(n^2)$

a) Let $k = C = 1$. Then if $n > k$,

$$f(n) = 1 + 2 + \cdots + n$$
$$\leq n + n + \cdots + n \quad \text{(since } 1, \ldots, n-1 < n\text{)}$$
$$= n^2$$
$$= C|n^2|$$

b) Let $k = 1$, $C = 1/4$. Then if $n > k$,

$$f(n) = 1 + \cdots + \lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n$$
$$\geq \lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n$$
$$\geq \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \quad \left(\text{since } \lceil n/2 \rceil < \lceil n/2 \rceil + 1, \ldots, n\right)$$
$$\geq n/2 + n/2 + \cdots + n/2 \quad \left(\text{since } \lceil n/2 \rceil \geq n/2\right)$$
$$\geq (n/2)(n/2) \quad \left(\text{since there are} \geq n/2 \text{ integers in the range } \lceil n/2 \rceil, \ldots, n\right)$$
$$= n^2/4 = C|n^2|$$

Scratch work:

$$1 + 2 + \cdots + \lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n$$

$$\geq \underbrace{\lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n}_{\geq n/2 \text{ terms each of which is } \geq n/2}$$

e.g.

$1 + 2 + \underbrace{3 + 4 + 5}$

$1 + 2 + 3 + \underbrace{4 + 5 + 6}$

$$\geq (n/2)(n/2)$$

$$= \frac{n^2}{4}$$