Announcements

Final exam: Thursday 12/19 1:30-4:30pm
4025 Campus Instructional Facility

Covers entire course

TWO reference sheets allowed (see policy email)

Review session: Tues. 12/17 10:00-11:30am Altgeld 147

Office hours: see email (but they may change)

Practice problems posted

Course evaluation: go.illinois.edu/ices-online

Final exam review

Partial list of topics:

Everything from midterms

(sets, functions, algorithms, induction, counting, probability, relations, graphs through 10.5)

Graphs (cont.)
- Shortest path problems
  - Weighted graphs
  - Dijkstra's algorithm
  - Travelling salesperson
- Planar graphs
  - Direct pf. of planarity/nonplanarity
  - Regions, degree, etc.
  - Euler's formula and consequences

Graph coloring

- Maps vs. graphs and their colorings
- Chromatic number
- Four-color theorem

Trees

- Definitions
- Properties
- Rooted trees, m-ary trees
- Applications: binary search trees, decision trees, game trees

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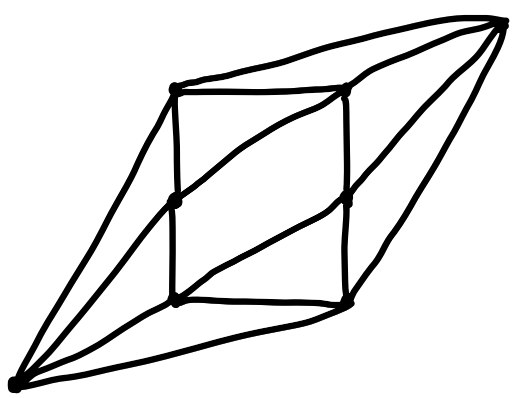
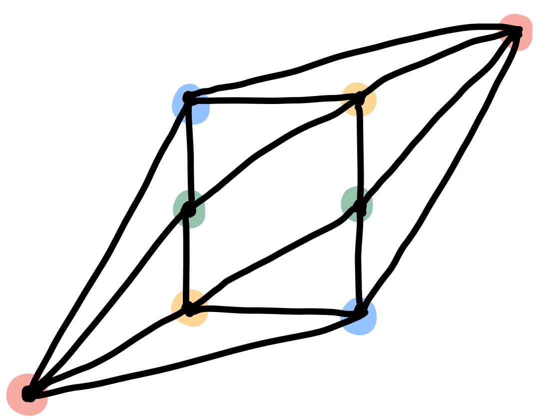
Examples:

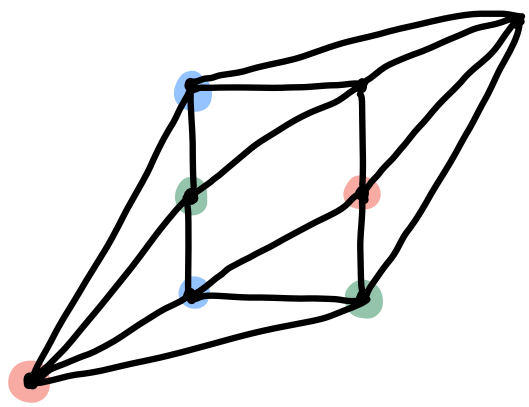
1) Determine the chromatic number of the following graph G



Sol'n: G can be 4-colored (see below)



However, no 3-coloring exists. Let red be the color of the bottom-left vertex. The top left vertex must have a different color; let that be blue. The vertex just below it must have a different color; let that be green. Using only those three colors, the following partial coloring is forced:



But then there is no valid color for either of the remaining vertices.

Therefore, $\chi(G) = 4$.

2) When rolling three dice, what is the conditional probability that the product is at least 10 given that the sum is 7

Sol'n: Possible ways to roll a sum of 7:

511 (3 orders) prod. is 5
421 (6 orders) prod. is 8
331 (3 orders) prod. is 9
322 (3 orders) prod. is 12

Number of ways to roll a sum of 7: 15
Num. of these ways where the prod. is ≥ 10: 3

Conditional prob.: $\frac{3}{15}$

3) Prove that $f(n) = n^2 e^n$ is $O(n^n)$

Pf: Let $C = 1$, $k = 10$. Then for $n > k$,

$$\frac{n}{e} > \frac{10}{e} > \frac{10}{3} > 3,$$

so $\frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n > 3^n$.

Next, we show that if $n \geq 3$, $n^2 < 3^n$ by induction.

Base case: If $n = 3$, $n^2 = 9 < 27 = 3^3$.

Inductive step: Suppose that $n \geq 3$ and $n^2 < 3^n$.

Then $\frac{n+1}{n} \leq \frac{4}{3} < 1.5$, so

$$(n+1)^2 = n^2 \cdot \left(\frac{n+1}{n}\right)^2 < n^2 \cdot (1.5)^2 = 2.25n^2 < 2.25 \cdot 3^n < 3^{n+1}.$$

Thus, we have shown that if $n \geq 3$, $n^2 < 3^n$ by induction.

Therefore, if $n > k$,

$$|f(n)| = n^2 e^n < n^2 \frac{n^n}{3^n} < n^n,$$

so f is $O(n^n)$. $\square$

4) Determine whether $f: \mathbb{Z} \to \mathbb{Z}$ is one-to-one (injective), onto (surjective), both, or neither.
(bijective)

a) $f(n) = \lfloor n/2 \rfloor$

b) $f(n) = n + (-1)^n$

c) $f(n) = 3n - 2$

Soln:

a) Not one-to-one since $f(0) = f(1) = 0$

Onto since if $y \in \mathbb{Z}$, $f(2y) = y$

b) One-to-one since $f(n) = n \pm 1$, so n and $f(n)$ always have the opposite parity

Let $x, y \in \mathbb{Z}$, $x \neq y$

- If x even, y odd, $f(x)$ is odd, $f(y)$ is even, so $f(x) \neq f(y)$
- If x odd, y even, $f(x)$ is even, $f(y)$ is odd, so $f(x) \neq f(y)$
- If x even, y even, $f(x) = x+1 \neq y+1 = f(y)$
- If x odd, y odd, $f(x) = x-1 \neq y-1 = f(y)$

Onto since if $z \in \mathbb{Z}$, let

$$x = \begin{cases} z+1, & \text{if } z \text{ even} \\ z-1, & \text{if } z \text{ odd} \end{cases}$$

- If z even, x is odd, so $f(x) = x - 1 = z$
- If z odd, x is even, so $f(x) = x + 1 = z$

Alternate method: f is bijective because it is invertible.

(Recall: g is the inverse of f if $g \circ f = f \circ g = id$)

- If x even, $(f \circ f)(x) = f(f(x)) = f(x+1) = x$
- If x odd, $(f \circ f)(x) = f(f(x)) = f(x-1) = x$

So $f^{-1} = f$ (!) and so x is invertible and therefore bijective (don't have to check both orders since $f^{-1} = f$)

c) One-to-one since if $3x-2 = 3y-2$, $3x = 3y$, so $x = y$

Not onto. $f(n) = 3n-2$ is always 2 less than a multiple of 3 i.e. the remainder when dividing $f(n)$ by 3 is always 1 i.e. $f(n)$ is always in the equiv. class $[1]$ in the equiv. rel'n $a \sim b$ if $a-b$ is a mult. of 3 (congruence class)

So in particular $0 \notin \text{range}(f)$, so f is not onto.