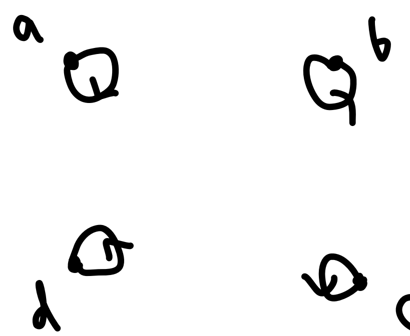
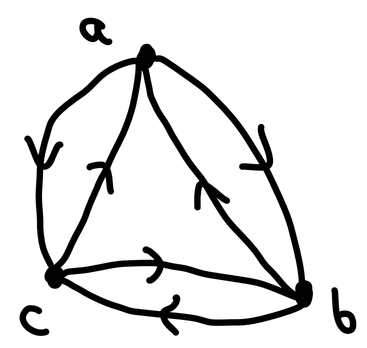
Recall: A relation from A to B is a subset of $A \times B$.
(a relation where each $a \in A$ appears exactly once is a function)

Properties:

- R is <u>reflexive</u> if aRa for all $a \in A$
- R is <u>symmetric</u> if whenever aRb, then bRa
- R is <u>antisymmetric</u> if whenever aRb and $a \neq b$, then $b \not R a$
- R is <u>transitive</u> if whenever aRb and bRc, then aRc

Class activity: Are the following rel'ns reflexive/symmetric/antisymmetric/transitive? Draw the corresponding matrix/digraph.

a) 

b) 

c)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Def: An <u>equivalence relation</u> on A is a rel'n on A which is reflexive, symmetric, and transitive

"a is equiv. to itself"

"if a is equiv. to b, then b is equiv to a"

"if a and b are equiv., and b and c are equiv, then a and c are equiv"

Often write $a \sim b$ for "a is equiv to b"

Def: The (maximal) subsets of A whose elts. are all equiv. are called the <u>equivalence classes</u> of A.

If $a \in A$, $[a] = \{b \in A \mid a \sim b\}$ is the equivalence class of a.

Ex 0: $A = \mathbb{Z}$. Let $\sim$ be the "parity" equivalence rel'n:

$a \sim b$ if and only if a and b are both even or both odd (same parity)

Reflexive: a has the same parity as itself

Symmetric: If a and b have the same parity, so do b and a

Transitive: If a & b have the same parity and so do b & c, both a and c have the same parity as b, and thus as each other

There are two equiv. classes:

$[0] = \{\text{even numbers}\}$ and $[1] = \{\text{odd numbers}\}$

Note that $\cdots = [-2] = [0] = [2] = [4] = \cdots$

representatives

Ex 1: $A = \mathbb{Z}$

$a \sim b$ means $a = b$ or $a = -b$

Reflexive: $a = a$

Symmetric: If $a = \pm b$, $b = \pm a$

Transitive: If $a = \pm b$, $b = \pm c$, then $a = \pm c$

Equivalence classes:

$[0] = \{0\}$

$[1] = \{-1, 1\}$

$[2] = \{-2, -2\}$

$\vdots$

Ex 7: $A = \mathbb{Z}$

aRb if $a-b$ is $0, 1,$ or -1

Reflexive, symmetric, but not transitive

e.g. $2R3$, $3R4$, but $2 \not R 4$

not an equiv. rel'n

Many (but not all) equiv. rel'ns are of the form:

$a \sim b$ means a and b share the same value of ____

Ex 0: parity

Ex 1: abs. value

Ex 4: $A = \{\text{binary strings}\} = \{\emptyset, 0, 1, 00, 01, \ldots\}$

$a \sim b$ if a and b have the same length ✓ equiv. rel'n

need to show explicitly

$[\emptyset] = \{\emptyset\}$ length 0

$[0] = [1] = \{0, 1\}$ length 1

$[00] = \{00, 01, 10, 11\}$ length 2

$[000] = \{000, 001, \ldots, 111\}$ length 3

Class activity (if time): Determine whether these are equiv. rel'ns ($A = \mathbb{Z}$)

a) $a \sim b$ if $a \mid b$

b) $a \sim b$ if $a \leq b$

c) $a \sim b$ if $a = b$

d) $a \sim b$ if $a - b$ is a mult. of 10

e) $a \sim b$ if $a - b$ is a mult. of 17

Every equivalence rel'n corresponds to a <u>set partition</u> (and vice-versa)

Def: A set partition of A is a set of subsets $A_1, A_2, \ldots$ s.t.

$A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \cdots = A$

i.e. every elt. of A is in exactly one A_i

The A_i correspond. to the equiv. classes of an equiv. rel'n.

equiv rel'n $\longleftrightarrow$ equiv. classes $\stackrel{?}{=}$ set partition

Ex 4 (cont.): The set partition corresp. to this equiv. rel'n is

$$A = A_0 \cup A_1 \cup \cdots$$

where

$A_i = \{\text{strings of length } i\}$

Ex 15: Let $A = \{\text{binary strings of length } 12\}$.

Set partition

$$\left.\begin{array}{l} A_{000} = \{\text{strings starting w/ } 000\} \\ A_{001} = \{\text{" " w/ } 001\} \\ \vdots \\ A_{111} = \{\text{" " w/ } 111\} \end{array}\right\} \text{set partition into 8 sets}$$

Corresp. equiv. rel'n:

$a \sim b$ if and only if a and b have the same first 3 digits

Ex 13: Let $A = A_1 \cup A_2 \cup A_3$ be a set partition with

$A_1 = \{1,2,3\}$ $A_2 = \{4,5\}$ $A_3 = \{6\}$

Class activity (if time): Find the corresp. equiv. rel'n.