Announcement

HW8 posted (due Sun. 11/2)

Recall: A relation from A to B is a subset of $A \times B$.
(a relation where each $a \in A$ appears exactly once is a function)

Properties:

- R is reflexive if aRa for all $a \in A$
- R is symmetric if whenever aRb, then bRa
- R is antisymmetric if whenever aRb and $a \neq b$, then $b \not R a$
- R is transitive if whenever aRb and bRc, then aRc

Operations:

- Complement: $\overline{R} = \{(a,b) \in A \times B \mid (a,b) \notin R\}$
- Inverse : $R^{-1} = \{(b,a) \mid (a,b) \in R\}$ relation from B to A
- Composition: If $R \subseteq A \times B$, $S \subseteq B \times C$, then

  $S \circ R = \{(a,c) \in A \times C \mid$ there exists $b \in B$ s.t. $(a,b) \in R, (b,c) \in S\}$
  relation from A to C

§9.3: Representing relations

Option 1: Use a matrix!

$A = \{a_1, a_2, \dots, a_m\}$

$B = \{b_1, \dots, b_n\}$

$R \subseteq A \times B$ relation

Then R can be represented by the matrix

$$M_R = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{otherwise} \end{cases}$$

Ex 1: $A = \{1,2,3\}$ $B = \{1,2\}$

$R = \{(2,1), (3,1), (3,2)\}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Class activity (Ex 3):

Let R be a relation on A s.t.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Find R. Is it reflexive? Symmetric? Antisymmetric?

If R & S are rel'ns on A, then

$$M_{R\cup S} = M_R \cup M_S$$

an entry is 1 if the corresp. entry in <u>either</u> M_R or M_S is 1

$$M_{R\cap S} = M_R \wedge M_S$$

same, but replace "either" with "<u>both</u>"

$$M_{S\circ R} = M_R \odot M_S$$

boolean prod: take matrix product, then change any pos. num. to 1

<u>note</u>: order reversed!

Ex: If $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R\cup S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad M_{R\cup S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_R M_S = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$

so $M_{S\circ R} = M_R \odot M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Option 2: Use a graph!

not the x,y kind of graph

Def: A <u>directed graph</u> or <u>digraph</u> consists of a set V of <u>vertices</u> and a set E of <u>edges</u>. Each edge goes from one vertex to another

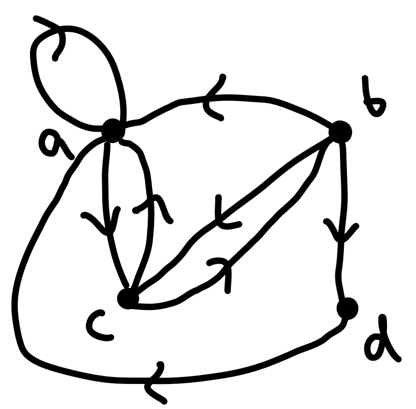
Let R be a relation on A. To make a digraph for R,

V: vertices labelled by elts. of A

E: edge going from a to b if $(a,b) \in R$

Ex 8: $A = \{a, b, c, d\}$

$R = \{(a,a), (a,c), (b,a), (b,c), (b,d), (c,a), (c,b), (d,a)\}$



Reflexive: loop at every vertex

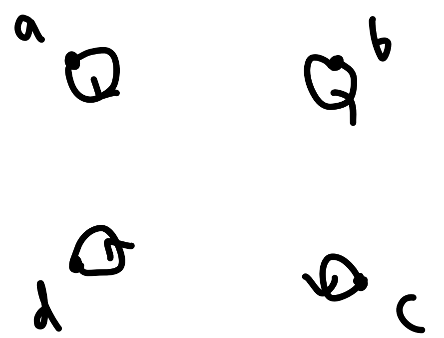
Symmetric: edges always come in opposite-direction pairs

Antisymmetric: edges never come in opposite-direction pairs

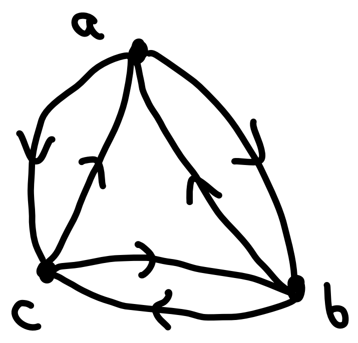
Transitive: For every length-2 path, there is a single edge w/ the same start and end

Class activity (if time): Are the following rel'ns reflexive/symmetric/antisymmetric/transitive?
Draw the corresponding matrix/digraph.

a)



b)



c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$