Announcements

No quiz next week

Midterm 2: Friday 10/25 in class

- Covers through Chapter 8
- Reference sheet allowed (one A4 sheet w/ writing on both sides)
- Fall policy email to come (practice problems, etc.)

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§8.6: Applications of Inclusion-Exclusion

Ex 1 (cont.): How many sol'ns does

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1, x_2, x_3 \in \mathbb{N}$ and

$x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 6$?

Sol'n: Let

$U = \{\text{all sol'ns}\}$

$A = \{\text{sol'ns w/ } x_1 \geq 4\}$

$B = \{\text{sol'ns w/ } x_2 \geq 5\}$

$C = \{\text{sol'ns w/ } x_3 \geq 7\}$

Want: $|U \setminus (A \cup B \cup C)| = |U| - |A \cup B \cup C|$

Sticks and stones:

$$|U| = \binom{11+(3-1)}{11} = \binom{13}{11} = 78$$

For A, let $y_1 = x_1 - 4$. Then $y_1, x_2, x_3 \in \mathbb{N}$ and $y_1 + x_2 + x_3 = 7$, so

$$|A| = \binom{7+(3-1)}{7} = \binom{9}{7} = 36$$

For B, let $y_2 = x_2 - 5$. Then $x_1, y_2, x_3 \in \mathbb{N}$ and $x_1 + y_2 + x_3 = 6$, so

$$|B| = \binom{6+(3-1)}{6} = \binom{8}{6} = 28$$

For C, let $y_3 = x_3 - 7$. Then $x_1, x_2, y_3 \in \mathbb{N}$ and $x_1 + x_2 + y_3 = 4$, so

$$|C| = \binom{4+(3-1)}{4} = \binom{6}{4} = 15$$

For $A \cap B$, $y_1, y_2, x_3 \in \mathbb{N}$, $y_1 + y_2 + x_3 = 2$, so $|A \cap B| = \binom{2+(3-1)}{2} = \binom{4}{2} = 6$

For $A \cap C$, $y_1, x_2, y_3 \in \mathbb{N}$, $y_1 + x_2 + y_3 = 0$,

so $|A \cap C| = \binom{0 + (3-1)}{0} = \binom{2}{0} = 1$

For $B \cap C$, $x_1, y_2, y_3 \in \mathbb{N}$, $x_1 + y_2 + y_3 = -1$,

so $|B \cap C| = 0$

$|A \cap B \cap C| = 0$ also, since $A \cap B \cap C \subseteq B \cap C$

Therefore, $|U \setminus (A \cup B \cup C)| = |U| - |A| - |B| - |C|$
$+ |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$
$= 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6$

Ex 2: How many surjective functions are there from $A \to B$ if $|A| = 6$, $|B| = 3$? What about if $|A| = m$, $|B| = n$?

Sol'n: Let $B = \{x, y, z\}$

$U = \{\text{functions } f: A \to B\}$

$P = \{f \in U \mid x \notin f(A)\}$ ← range of f

$Q = \{f \in U \mid y \notin f(A)\}$

$R = \{f \in U \mid z \notin f(A)\}$

Want: $|U \setminus (P \cup Q \cup R)|$

$|U| = 3^6 \quad |P| = |Q| = |R| = 2^6$

$|P \cap Q| = |P \cap R| = |Q \cap R| = 1^6 \quad |P \cap Q \cap R| = 0$

So $|U \setminus (P \cup Q \cup R)| = |U| - |P \cup Q \cup R|$

$= |U| - |P| - |Q| - |R| + |P \cap Q| + |P \cap R| + |Q \cap R| - |P \cap Q \cap R|$

$= 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 - 0$

$= 729 - 192 + 3 = 540$

Similarly, if $|A| = m$, $|B| = n$, there are

$$\binom{n}{0} n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \dots + (-1)^{n+1}\binom{n}{n-1} 1^m$$

choose the
2 elts. from
B to exclude

Def: A <u>derangement</u> is a permutation of objects that leaves no object in its initial position

e.g. 21453 ✓ 3142$\underline{5}$ x 54$\underline{3}$21 x $\underline{1}\underline{2}\underline{3}\underline{4}\underline{5}$ x

Ex 4: What is the number of derangements of n objects?

Sol'n: $U = \{\text{all permutations of } n \text{ objects}\}$

$A_i = \{\text{permutations where } i \text{ is in the } i\text{th spot}\}$

Want: $|\overline{A_1 \cup \cdots \cup A_n}|$

$|U| = n!$

$|A_i| = (n-1)!$ since the ith spot is fixed

$|A_i \cap A_j| = (n-2)!$ since the ith, jth spots are fixed

$\vdots$

$$|\overline{A_1 \cup \cdots \cup A_n}| = |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

$$= n! - \sum_i (n-1)! + \sum_{i<j} (n-2)! + \cdots + (-1)^n 0!$$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^n \binom{n}{n} 0!$$

$$= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! + \cdots + (-1)^n \frac{n!}{0!n!} 0!$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Follow-up: what is the probability that a randomly chosen permutation of a set of size n is a derangement?

$E = \{\text{derangements}\}$ $\quad S = \{\text{all permutations}\}$

$$p(E) = \frac{|E|}{|S|} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

Something to think about if you've seen Taylor series!
what happens to this probability if n gets very large?