Announcements

Quiz 5 today!

Monday's lecture video posted

Office hour today instead of Monday

HW 7 posted (due Sunday 10/20)

Midterm 2: Friday, 10/25 in class

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§8.5: Inclusion - Exclusion

Recall: subtraction principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$x \in A$, $x \in B$	1	$=$	$1 + 1 -$	1	
$x \in A$, $x \notin B$	1	$=$	$1 + 0 -$	0	
$x \notin A$, $x \in B$	1	$=$	$0 + 1 -$	0	
$x \notin A$, $x \notin B$	0	$=$	$0 + 0 -$	0	

Need to count every elt. of $A \cup B$ exactly once

Class activity: Do the same thing w/ three sets

$$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$$

Inclusion-Exclusion:

3 sets:

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C| + |A\cap B\cap C|$$

4 sets:

$$\begin{aligned}|A\cup B\cup C\cup D| = {} & |A|+|B|+|C|+|D|-|A\cap B|-|A\cap C| \\ & -|A\cap D|-|B\cap C|-|B\cap D|-|C\cap D| \\ & +|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D| \\ & +|B\cap C\cap D|-|A\cap B\cap C\cap D|\end{aligned}$$

n sets:

$$|A_1\cup\cdots\cup A_n| = \sum_{1\le i\le n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1<i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n+1}|A_1\cap\cdots A_n|$$

Reason this works:

Binomial theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = \underbrace{\binom{n}{0}}_{1} - \binom{n}{1} + \binom{n}{2} + \cdots (-1)^n \binom{n}{n}$$

$$1 = \underbrace{\binom{n}{1}}_{\#\text{terms } |A_i|} - \underbrace{\binom{n}{2}}_{\#\text{terms } |A_i \cap A_j|} + \underbrace{\binom{n}{3}}_{\#\text{terms } |A_i \cap A_j \cap A_k|} - \cdots + (-1)^{n+1} \underbrace{\binom{n}{n}}_{\#\text{terms } |A_1 \cap \cdots \cap A_n|}$$

Ex 4: $|A| = 1232$ $\quad |A \cap B| = 103$

$|B| = 879$ $\quad |A \cap C| = 23$

$|C| = 114$ $\quad |B \cap C| = 14$

$|A \cup B \cup C| = 2092$ Find $|A \cap B \cap C|$

Sol'n: Inclusion-exclusion:

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |A \cap B \cap C|$$

$$= 2085 + |A \cap B \cap C|$$

So $|A \cap B \cap C| = 2092 - 2085 = 7$

§8.6 Ex 1: How many sol'ns does

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1, x_2, x_3 \in \mathbb{N}$ and

$x_1 \le 3$, $x_2 \le 4$, $x_3 \le 6$?

Sol'n: Let

$U = \{\text{all sol'ns}\}$

$A = \{\text{sol'ns w/ } x_1 \ge 4\}$

$B = \{\text{sol'ns w/ } x_2 \ge 5\}$

$C = \{\text{sol'ns w/ } x_3 \ge 7\}$

Want: $|U \setminus (A \cup B \cup C)| = |U| - |A \cup B \cup C|$

Sticks and stones:

$$|U| = \binom{11 + (3-1)}{11} = \binom{13}{11} = 78$$

For A, let $y_1 = x_1 - 4$. Then $y_1, x_2, x_3 \in \mathbb{N}$ and $y_1 + x_2 + x_3 = 7$, so

$$|A| = \binom{7 + (3-1)}{7} = \binom{9}{7} = 36$$

For B, let $y_2 = x_2 - 5$. Then $x_1, y_2, x_3 \in \mathbb{N}$ and $x_1 + y_2 + x_3 = 6$, so

$$|B| = \binom{6+(3-1)}{6} = \binom{8}{6} = 28$$

For C, let $y_3 = x_3 - 7$. Then $x_1, x_2, y_3 \in \mathbb{N}$ and $x_1 + x_2 + y_3 = 4$, so

$$|C| = \binom{4+(3-1)}{4} = \binom{6}{4} = 15$$

For $A \cap B$, $y_1, y_2, x_3 \in \mathbb{N}$, $y_1 + y_2 + x_3 = 2$,

so $|A \cap B| = \binom{2+(3-1)}{2} = \binom{4}{2} = 6$

For $A \cap C$, $y_1, x_2, y_3 \in \mathbb{N}$, $y_1 + x_2 + y_3 = 0$,

so $|A \cap C| = \binom{0+(3-1)}{0} = \binom{2}{0} = 1$

For $B \cap C$, $x_1, y_2, y_3 \in \mathbb{N}$, $x_1 + y_2 + y_3 = -1$,

so $|B \cap C| = 0$

$|A \cap B \cap C| = 0$ also, since $A \cap B \cap C \subseteq B \cap C$

Therefore, $|U \setminus (A \cup B \cup C)| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$= 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6$