Announcements

This recording replaces Monday's lecture
(no in-person class Monday)
Office hour moved to Wednesday

HW 7 posted (due Sunday 10/20)

Quiz 5: Wednesday

Midterm 2: Friday, 10/25 in class

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§8.2: Solving Linear recurrence relations

Recall: A sequence is an infinite list of numbers

$$a_1, a_2, a_3, \ldots$$

doesn't need to start w/ a_1

A recurrence relation is a formula for a_n in terms of (some of) $a_1, a_2, \ldots, a_{n-1}$.

Given a recurrence rel'n and some initial condition(s) (value of at least a_1) we try to solve the recurrence rel'n by giving an explicit formula (not a recurrence rel'n) for a_n.

Today we'll learn how to solve one particular kind of recurrence rel'n

Def: A linear homogeneous recurrence rel'n of deg. k w/ constant coefficients is a recurrence rel'n of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$.

Ex: $f_n = f_{n-1} + f_{n-2}$ ✓ linear homog. of deg 2

$H_n = 2H_{n-1} + 1$ ✗ not linear homog.

✓ but linear inhomog.

$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0$ ✗ not linear homog.

Ex: $a_n = c a_{n-1}$, $a_1 = c$ linear homog. of deg 1

Then, $a_2 = c a_1 = c^2$

$a_3 = c a_2 = c^3$

$\vdots$

$a_n = c^n$

Let's look for sol'ns of the form

$a_n = r^n$ for any linear homog. recurrence rel'n

Need to find r

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad a_n = r^n$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Divide by r^{n-k}, and move everything to the LHS:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

This is called the <u>characteristic eqn.</u> for $\{a_n\}$

We can only have $a_n = r^n$ if r is a root of the characteristic eqn.

In fact,

Thm: Suppose that

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$

Then, $\{a_n\}$ is a sol'n of the recurrence rel'n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for some values $\alpha_1, \ldots, \alpha_k$

Ex 4: $f_n = f_{n-1} + f_{n-2}$ degree $k=2$ $c_1 = c_2 = 1$

So the char. eqn. is

$$r^2 - r - 1 = 0$$

Roots: $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

So must have

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$= \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ for some } \alpha_1, \alpha_2 \in \mathbb{R}.$$

To find α_1, α_2, plug in initial conds.

$$f_1 = 1, \; f_2 = 1, \; f_0 = 0$$

$f_0 = 0$, so $0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0$

$$= \alpha_1 + \alpha_2$$

$f_1 = 1$, so $1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$

Sol'n: $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$

Therefore,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex 3: Solve the recurrence rel'n

$$a_n = a_{n-1} + 2a_{n-2}, \qquad a_0 = 2,\ a_1 = 7$$

Sol'n: Char. eqn: $r^2 - r - 2 = 0$

Roots: $r_1 = 2$, $r_2 = -1$

So $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

Plug in to find α_1, α_2:

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1(2) + \alpha_2(-1) = 2\alpha_1 - \alpha_2$

$\alpha_1 = 3$, $\alpha_2 = -1$

So $a_n = 3 \cdot 2^n - (-1)^n$

Thm (deg. 2 w/ repeated roots):

Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has a repeated root r

Then, $\{a_n\}$ is a sol'n of the recurrence rel'n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = (\alpha + \beta n) r^n \quad \text{for some values } \alpha, \beta$$

Ex: $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = a_1 = 3$

Char. eqn.: $r^2 - 4r + 4 = 0$

(Double) root: $r = 2$

So $a_n = (\alpha + \beta n) 2^n$

Plug in to match initial conds:

$$a_0 = 3 = \alpha$$

$$a_1 = 3 = (\alpha + \beta) 2$$

$$\alpha = 3, \ \beta = -\frac{3}{2}$$

$$a_n = \left(3 - \frac{3}{2} n\right) 2^n$$

One last modification: let's turn this <u>inhomogeneous</u> by adding a function of n:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

Thm: Suppose you can find <u>any</u> particular sol'n $\{a_n^{(p)}\}$ to this recurrence rel'n

Then every sol'n is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a sol'n to the homog. eqn:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Ex 12: Find all sol'ns to the recurrence rel'n

$$a_n = 6a_{n-1} - 9a_{n-2} + n3^n$$

Sol'n:

Homog. rec. rel'n: $a_n = 6a_{n-1} - 9a_{n-2}$

Char. eqn:

$$r^2 - 6r + 9 = 0$$

$r = 3$ (double root!)

So the sol'ns to the homog. eqn. are:

$$a_n^{(h)} = (\alpha + \beta n) 3^n$$

Now we need a particular sol'n:

Theorem 6 in Rosen $\rightarrow a_n^{(p)}$ has the form

$$a_n^{(p)} = n^2 (p_1 n + p_0) 3^n$$

Solve for p_0 and p_1 by plugging into the recurrence rel'n:

$$a_n = 6a_{n-1} - 9a_{n-2} + n 3^n$$

$$n^2(p_1 n + p_0) 3^n = 6(n-1)^2 (p_1(n-1) + p_0) 3^{n-1}$$
$$- 9(n-2)^2 (p_1(n-2) + p_0) 3^{n-2} + n 3^n$$

Match

$n 3^n$ coeffs: $0 = 1 - 6p_1$

3^n coeffs: $0 = 6p_1 - 2p_0$

$p_1 = \frac{1}{6}, \ p_0 = \frac{1}{2}$

So $a_n^{(p)} = n^2\left(\frac{1}{6} n + \frac{1}{2}\right) 3^n$

General sol'n: $a_n = a_n^{(h)} + a_n^{(p)}$

$$= (\alpha + \beta n) 3^n + n^2\left(\frac{1}{6} n + \frac{1}{2}\right) 3^n$$