Quiz 4 today!

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§ 7.3: Bayes Theorem

Recall: conditional probability:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Bayes' Theorem: Assume $p(E), p(F) > 0$. Then,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}$$

Pf: By definition,

$$p(E|F) = \frac{p(E \cap F)}{p(F)} \quad \text{and} \quad p(F|E) = \frac{p(E \cap F)}{p(E)},$$

so $p(E \cap F) = p(E|F)\,p(F)$, and combining this w/ the 2nd eqn,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)}. \qquad \square$$

Important note: often, $p(E)$ is unknown, so Rosen gives the alternate version:

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F})}$$

This is equivalent to our version, since

$$p(E|F)p(F) + p(E|\overline{F})p(\overline{F}) = p(E \cap F) + p(E \cap \overline{F}) = p(E)$$

sum rule, since $E \cap F$ and $E \cap \overline{F}$ have no overlap

Ex 2: Suppose one person in 100,000 has a particular rare disease. There is a diagnostic test which is correct

- 99% of the time, when given to a person w/ the disease
- 99.5% of the time, when given to a person w/out the disease

Find the probability that a person who tests positive actually has the disease.

Sol'n: E: tests positive, F: has the disease

Want: $p(F|E)$.

$p(F) = \frac{1}{100000} = 0.00001$ $\quad p(\overline{F}) = 0.99999$

$p(E|F) = 0.99$ $\quad p(E|\overline{F}) = 1 - 0.995 = 0.005$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)} = \frac{p(E|F)\,p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

$$= \frac{0.99 \cdot 0.00001}{0.99 \cdot 0.00001 + 0.005 \cdot 0.99999} = 0.002 = 0.2\%$$

Even though the test is very good, almost all of the positive tests are false positives!

Ex 1 (Class activity if time):

Two boxes

Box 1: 2 Green balls, 7 Red balls

Box 2: 4 Green balls, 3 Red balls

We

- Choose a box at random ($p(\text{Box } 1) = 0.5$)
- Choose a ball at random from that box (equal prob for each ball in the box)

If we select a Red ball, what is the probability it came from the first box?

Sol'n: E: Red ball $\quad \overline{E}$: Green ball

F: Box 1 $\quad \overline{F}$: Box 2

Want: $p(F \mid E)$

$p(E|F) = \frac{7}{9} \quad p(E|\overline{F}) = \frac{3}{7}$

$p(F) = p(\overline{F}) = \frac{1}{2}$

$$p(E) = p(E|F)\,p(F) + p(E|\overline{F})\,p(\overline{F}) = \frac{7}{9}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{2} = \frac{38}{63}$$

By Bayes' Theorem,

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E)} = \frac{\frac{7}{9}\cdot\frac{1}{2}}{38/63} = \frac{49}{76} \approx 0.645$$