§6.3: Permutations and Combinations

Ex I: How many ways are there to order 3 people in a line?

ABC BAC CAB
ACB BCA CBA
} permutations of $\{A, B, C\}$

Ans: $3 \cdot 2 \cdot 1 = 6$ (from prod. rule)

Ex II: How many ways are there to order 2 out of 5 people in a line?

AB BA CA DA EA
AC BC CB DB EB
AD BD CD DC EC
AE BE CE DE ED
} 2-permutations of $\{A, B, C, D, E\}$

Ans: $5 \cdot 4 = 20$

Def: A permutation of a set is an ordered arrangement of its elts. An r-permutation is an ordered arrangement of r of its elts.

The number of r-permutations of a set of card. n is

$$P(n,r) = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}} = \frac{n!}{(n-r)!} \qquad (P(n,n) = n!)$$

"n permute r"

Ex 7: How many permutations of the letters ABCDEFGH contain the string ABC?

e.g. DGHABCFE

Ans: Just take permutations of {ABC, D, E, F, G, H}

$P(6,6) = 6! = 5040$

Ex III: How many committees of 2 students can be formed out of a total of 5 students?

AB BC CE
AC BD DE
AD BE
AE CD

10 total

Note: here AB and BA represent the same committee

Half as many as in Ex II since there are 2! orders for each committee.

Ex IV: There are $P(10,3) = 10 \cdot 9 \cdot 8 = 720$ ways to order 3 out of 10 people. But,

ABC BCA
ACB CAB
BAC CBA
} same committee

BFG FGB
BGF GBF
FBG GFB
} same committee

So there are only $\frac{720}{3!} = 120$ 3-person committees out of 10 people

Def: A k-combination of a set is an unordered arrangement of k of its elts. i.e. an k-elt. subset!

The number of k-combinations of a set of card. n is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{P(n,k)}{k!}$$

"n choose k" ← binomial coefficients

Ex 11: How many ways are there to choose 5 (unordered) cards from a 52-card deck?

$$\text{Ans: } \binom{52}{5} = \frac{52!}{5!\ 47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46\cdots}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 47\cdot 46\cdots}$$

$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2598960$$

How many ways to choose 47 cards?

$$\text{Ans: } \binom{52}{47} = \frac{52!}{47!\ 5!} = \binom{52}{5} =$$

$$\binom{n}{k} = \binom{n}{n-k} \text{ always}$$

Class activity:

a) How many 3-elt subsets of $\{A, B, C, D, E, F\}$ are there?

b) How many contain A?

c) How many don't contain A?

d) Can you express the above 3 quantities as binom. coeffs. $C(n,r)$?

e) Does the above say anything about those binom. coeffs.?

§6.4: Binomial coeffs. and identities

Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{0}{0}$$

$$\binom{1}{0}\ \binom{1}{1}$$

$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$

$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$

$$\vdots \qquad \vdots \qquad \ddots$$

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$\ddots \qquad \vdots \qquad \ddots$$

Pascal's triangle!

Class activity (if time):

Expand $(x+y)^3$ and $(x+y)^4$ by FOIL-ing