Problem §8.6 - 3: How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2, and x_3 are nonnegative integers less than 6?

Solution. Let P_1 be the property that $x_1 \geq 6$, P_2 be the property that $x_2 \geq 6$, and P_3 be the property that $x_3 \geq 6$. To count the number of solutions to the equation $x_1 + x_2 + x_3 = 13$ where x_1, x_2, and x_3 are nonnegative integers less than 6, we want to compute $N(P_1', P_2', P_3')$ (see Rosen for this notation) using the principle of inclusion-exclusion. We know that

$$N(P_1', P_2', P_3') = N - (N(P_1) + N(P_2) + N(P_3)) + (N(P_1, P_2) + N(P_2, P_3) + N(P_1, P_3)) - N(P_1, P_2, P_3).$$

Recall that N is the number of all solutions to $x_1 + x_2 + x_3 = 13$ where x_1, x_2 and x_3 can be any nonnegative integers. From many similar problems in Chapter 6, we know that

$$N = \left(\!\!\binom{3}{13}\!\!\right) = \binom{3+13-1}{13} = \binom{15}{13}.$$

We can also observe that by symmetry, $N(P_1) = N(P_2) = N(P_3)$. Again using reasoning that we developed in Chapter 6, we know that

$$N(P_i) = \begin{pmatrix} \text{number of solutions} \\ \text{with } x_i \geq 6 \end{pmatrix} = \left(\!\!\binom{3}{13-6}\!\!\right) = \left(\!\!\binom{3}{7}\!\!\right) = \binom{3+7-1}{7} = \binom{9}{7}.$$

Similarly, $N(P_1, P_2) = N(P_2, P_3) = N(P_1, P_3)$ and

$$N(P_i, P_j) = \begin{pmatrix} \text{number of solutions} \\ \text{with } x_i, x_j \geq 6 \end{pmatrix} = \left(\!\!\binom{3}{13-2\cdot 6}\!\!\right) = \left(\!\!\binom{3}{1}\!\!\right) = \binom{3+1-1}{1} = \binom{3}{1}.$$

Finally, we can observe that $N(P_1, P_2, P_3) = 0$ because $3 \cdot 6 = 18 > 13$ and therefore it's not possible to have a solution where $x_1, x_2, x_3 \geq 6$.

Substituting for these values, we find that

$$\begin{aligned} N(P_1', P_2', P_3') &= N - (N(P_1) + N(P_2) + N(P_3)) + (N(P_1, P_2) + N(P_2, P_3) + N(P_1, P_3)) - N(P_1, P_2, P_3) \\ &= \binom{15}{13} - \binom{3}{1} \cdot \binom{9}{7} + \binom{3}{2} \cdot \binom{3}{1} - 0 \\ &= \binom{15}{13} - 3 \cdot \binom{9}{7} + 3^2 \\ &= 6 \end{aligned}$$

Because this is such a small number, we could actually explicitly verify it. In general, though, trying to explicitly find the number of solutions would be a very cumbersome and ineffecient process! □

Problem §8.6 - 10: In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?

Solution. This question is essentially asking us to count the number of onto functions from a set with eight elements (the distinguishable balls) to a set with three elements (the distinguishable urns). From Theorem 1 in §8.6, we know that there are

$$3^8 - \binom{3}{1} \cdot 2^8 + \binom{3}{2} \cdot 1^8 = 5796$$

such functions and therefore 5796 ways to distribute eight distinct balls into three distinct urns if each urn must contain at least one ball. □

Problem §8.6 - 14: What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

Solution. We wish to find the probability of the event, E, that nobody receives their own hat back. Our sample space, S, is the set of all possible outcomes - i.e., all possible ways to return 10 hats to 10 people. Clearly, $|S| = 10!$.

To find $|E|$, we simply apply our formula for D_n, which appears as Theorem 2 from §8.6, with $n = 10$. Doing so, we find

$$|E| = 10! \cdot \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right]$$

and therefore

$$\begin{aligned} p(E) = \frac{|E|}{|S|} &= \frac{10! \cdot \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right]}{10!} \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!} \\ &\approx 0.37 \end{aligned}$$

□

Problem §9.1 - 6(a-f): Determine whether the relation R on the set of all real number is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

(a) $x + y = 0$.

(b) $x = \pm y$.

(c) $x - y$ is a rational number.

(d) $x = 2y$.

(e) $xy \geq 0$.

(f) $xy = 0$.

Solution. (a) This relation is not reflexive - for example, $(1, 1) \notin R$ because $1 + 1 \neq 0$. It is symmetric: because $x + y = y + x$, we know that $x + y = 0$ if and only if $y + x = 0$. As such, $(x, y) \in R$ implies $(y, x) \in R$. It is not antisymmetric - for example, $(-1, 1) \in R$ and $(1, -1) \in R$ because $-1+1 = 0$ and $1+(-1) = 0$ but $1 \neq -1$. It is not transitive - for example, $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$.

(b) This relation is reflexive, because $x = \pm x$ (where we choose the plus sign) so $(x, x) \in R$. It is also symmetric: if $(x, y) \in R$, then $x = \pm y$ so $y = \pm x$ and $(y, x) \in R$. It is not antisymmetric - for example, $(1, -1) \in R$ and $(-1, 1) \in R$ because $1 = \pm(-1)$ and $-1 = \pm 1$ (where we choose the minus signs) but $1 \neq -1$. It is, however, transitive. If $(x, y) \in R$ and $(y, z) \in R$, then $x = \pm y$ and $y = \pm z$. Hence, $x = \pm(\pm z) = \pm z$ and $(x, z) \in R$.

(c) This relation is reflexive because $x - x = 0$ is rational, hence $(x, x) \in R$, for all real numbers x. It is also symmetric because if $x - y$ is rational, then so is $y - x = -(x - y)$. It is not, however, antisymmetric - for example, $1 - (-1) = 2$ and $(-1) - 1 = -2$ are both rational so we have $(1, -1), (-1, 1) \in R$ but $1 \neq -1$. It is transitive. If $(x, y), (y, z) \in R$ then $x - y$ and $y - z$ are both rational. Then $x - z = (x - y) + (y - z)$ must also be rational, since it's the sum of two rational numbers, and we have $(x, z) \in R$.

(d) This relation is not reflexive. For any $x \neq 0$, $x \neq 2x$ and therefore $(x, x) \notin R$. It is also not symmetric, since $(2, 1) \in R$ but $(1, 2) \notin R$. It is antisymmetric! To see this, suppose $(x, y) \in R$ and $(y, x) \in R$ so $x = 2y$ and $y = 2x$. Then $y = 2x = 4y$ which implies $y = 0$ and therefore $x = 0 = y$. The relation is not transitive - for exampe, $(4, 2) \in R$ and $(2, 1) \in R$ but $(4, 1) \notin R$.

(e) This relation is reflexive; because $x^2 \geq 0$ for all real numbers x, we always have $(x, x) \in R$. It is also symmetric because multiplication is associative so $xy \geq 0$ implies $yx \geq 0$. It is not antisymmetric - for example, $(2, 0) \in R$ and $(0, 2) \in R$ but $0 \neq 2$. It is also not transitive - for example, $(1, 0) \in R$ and $(0, -2) \in R$ but $(1, -2) \notin R$.

(f) This relation is not reflexive - for example, $(1, 1) \notin R$ because $1 \cdot 1 \neq 0$. It is symmetric, not antisymmetric, and not transitive by the same reasoning as in part (e). In fact, the same counterexamples even work for this relation!

□

Problem §9.1 - 15: Can a relation on a set be neither reflexive nor irreflexive?

Solution. Recall that a relation R on a set A is *irreflexive* if for every $a \in A$, $(a, a) \notin R$ (i.e., no element of A is related to itself by R).

A relation can be neither reflexive or irreflexive! As an example, let $A = \{1, 2, 3\}$ and consider

$$R = \{(2, 2), (1, 3), (3, 3)\}.$$

This relation is not reflexive because it does not contain $(1, 1)$. It is not irreflexive either, however, because it *does* contain $(2, 2)$ and $(3, 3)$.

In general, a relation R on a set A will be neither reflexive nor irreflexive if it contains the ordered pairs (a, a) for $a \in B$ where B is some *non-empty proper subset* of A - i.e., it contains (a, a) for at least one element of A, but not *every* element. □

Problem §9.1 - 22: Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation also be asymmetric?

Solution. Recall that a relation R on a set A is *asymmetric* if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

A relation that is asymmetric must also be antisymmetric, since the hypothesis for being antisymmetric ("suppose $(a, b) \in R$ and $(b, a) \in R$...") is false and so the condition is trivially satisfied. An antisymmetric relation need not also be asymmetric, however. For example, let $A = \{a, b\}$ and consider $R = \{(a, a), (a, b), (b, b)\}$. This relation is trivially antisymmetric, but it is not asymmetric because $(a, a) \in R$ clearly implies $(a, a) \in R$ and likewise with (b, b).

In general, a relation R is asymmetric if and only if it is antisymmetric and irreflexive. □

Problem §9.1 - 26: Let R be the relation $R = \{(a, b) : a < b\}$ on the set of integers. Find

(a) R^{-1}.

(b) $\overline{R}$.

Solution. Recall that if R is a relation from a set A to a set B, then

$$R^{-1} = \{(b, a) : (a, b) \in R\},$$
$$\overline{R} = \{(a, b) : (a, b) \notin R\}.$$

Using the relation R given in the problem statement, we can then observe that

(a) $R^{-1} = \{(b,a) : (a,b) \in R\} = \{(b,a) : a < b\}$, which we can write more simply as

$$R^{-1} = \{(a,b) : a > b\}.$$

(b) $\overline{R} = \{(a,b) : (a,b) \notin R\} = \{(a,b) : a \not< b\}$, which we can write more simply as

$$\overline{R} = \{(a,b) : a \geq b\}.$$

□

Problem §9.3 - 2(a,b): Represent each of these relations on $\{1,2,3,4\}$ with a matrix (with the elements of this set listed in increasing order).

(a) $\{(1,2),\ (1,3),\ (1,4)\ ,(2,3),\ (2,4),\ (3,4)\}$

(b) $\{(1,1),\ (1,4),\ (2,2),\ (3,3),\ (4,1)\}$

Solution. From the definition of the matrix representation of a relation, we know that position (i,j) of the matrix will have a 1 if the pair (i,j) is in the relation and a 0 if the pair (i,j) is not in the relation. Indexing the rows and columns of our matrices with $\{1,2,3,4\}$ in increasing order, we find:

(a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

□

Problem §9.3 - 14(a-d): Let R_1 and R_2 be relations on a set A represented by the matrices

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices that represent

(a) $R_1 \cup R_2$.

(b) $R_1 \cap R_2$.

(c) $R_2 \circ R_1$.

(d) $R_1 \circ R_1$.

Solution. (a) Recall that we find $M_{R_1 \cup R_2}$ by taking the join of the matrices M_{R_1} and M_{R_2}. Hence,

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Recall that we find $M_{R_1 \cap R_2}$ by taking the meet of the matrices M_{R_1} and M_{R_2}. Hence,

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Using the appropriate Boolean product, we compute

$$\begin{aligned} M_{R_2 \circ R_1} &= M_{R_1} \odot M_{R_2} \\ &= \begin{bmatrix} (0\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & (0\wedge 1)\vee(1\wedge 1)\vee(0\wedge 1) & (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) \\ (1\wedge 0)\vee(1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 1)\vee(1\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(1\wedge 1) \\ (1\wedge 0)\vee(0\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(0\wedge 1)\vee(0\wedge 1) & (0\wedge 0)\vee(0\wedge 1)\vee(0\wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

(d) Using the appropriate Boolean product, we compute

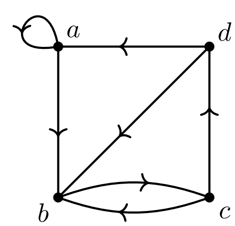
$$\begin{aligned} M_{R_1 \circ R_1} &= M_{R_1} \odot M_{R_1} \\ &= \begin{bmatrix} (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) & (0\wedge 1)\vee(1\wedge 1)\vee(0\wedge 0) & (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) \\ (1\wedge 0)\vee(1\wedge 1)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 1)\vee(1\wedge 0) & (1\wedge 0)\vee(1\wedge 1)\vee(1\wedge 0) \\ (1\wedge 0)\vee(0\wedge 1)\vee(0\wedge 1) & (1\wedge 1)\vee(0\wedge 0)\vee(0\wedge 0) & (1\wedge 0)\vee(0\wedge 1)\vee(0\wedge 0) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

□

Problem §9.3 - 22: Draw the directed graph that represents the relation

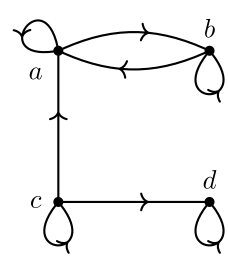
$$\{(a,a),\ (a,b),\ (b,c),\ (c,b),\ (c,d),\ (d,a),\ (d,b)\}.$$

Solution. We can draw the directed graph representation of this relation by drawing vertices with labels from the set $\{a,b,c,d\}$ and drawing an arrow $i \to j$ whenever (i,j) is in the relation. Doing so, we draw



□

Problem §9.3 - 26: List the ordered pairs in the relations represented by the directed graph



Solution. This is the directed graph representation of the relation

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, c), (c, d), (d, d)\}$$

on the set $\{a, b, c, d\}$. $\square$