Problem §8.1 - 12:

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three at a time.

(b) What are the initial conditions?

(c) In how many ways can this person climb a flight of eight stairs?

Solution. Let a_n denote the number of ways to climb n stairs.

(a) For $n \geq 3$, we can break the set of ways to climb n stairs into the following cases:

1. We can start by climbing a single stair, then climb the remaining $n-1$ stairs in one of a_{n-1} ways.
2. We can start by climbing two stairs at once, then climb the remaining $n-2$ stairs in one of a_{n-2} ways.
3. We can start by climbing three stairs at once, tehn climb the remaining $n-3$ stairs in one of a_{n-3} ways.

These cases are mutually exclusive and together include all possible methods of climbing n stairs. From this breakdown, we can write a recurrence relation for a_n as

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

(b) Next, we can determine initial conditions. There is only one way to climb no stairs (i.e., do nothing), so $a_0 = 1$. Likewise, there is only one way to climb a single stair so $a_1 = 1$. A set of two stairs can be climbed in two ways (either taking one step twice or two steps once), so $a_2 = 2$.

(c) This part asks us to compute a_8. Using repeated applications of the recurrence relation, we find

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 = 2 + 1 + 1 = 4 \\
a_4 &= a_3 + a_2 + a_1 = 4 + 2 + 1 = 7 \\
a_5 &= a_4 + a_3 + a_2 = 7 + 4 + 2 = 13 \\
a_6 &= a_5 + a_4 + a_3 = 13 + 7 + 4 = 24 \\
a_7 &= a_6 + a_5 + a_4 = 24 + 13 + 7 = 44 \\
a_8 &= a_7 + a_6 + a_5 = 44 + 24 + 13 = 81
\end{aligned}$$

□

Problem §8.1 - 20: A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

(a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

(b) In how many different ways can the driver pay a toll of 45 cents?

Solution. Let a_n denote the number of ways to pay a toll of $5n$ cents. This notation is appropriate for the problem because it's only possible to pay a toll that is a multiple of 5 cents if the driver is using only nickles and dimes, which are respectively worth 5 and 10 cents.

(a) We can break the number of ways to pay a toll of $5n$ cents into two cases:

1. The driver could first use a nickel, then pay the remaining $5n - 5 = 5(n-1)$ cents using some combination of nickels and dimes.
2. The driver could first use a dime, then pay teh remaining $5n - 10 = 5(n-2)$ cents using some combination of nickels and dimes.

Because these cases are mutually exclusive and together describe every way that the driver could pay using nickels and dimes, this suggests the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 2.$$

The appropriate initial conditions are $a_0 = a_1 = 1$, since there is exactly one way to pay no toll (do nothing) and one way to pay a toll of 5 cents (use one nickel).

(b) We're asked to find the number of ways to pay a toll of $45 = 5(9)$ cents, i.e. to compute a_9. Iterating,

$$\begin{aligned}
a_2 &= a_1 + a_0 = 1 + 1 = 2 \\
a_3 &= a_2 + a_1 = 2 + 1 = 3 \\
a_4 &= a_3 + a_2 = 3 + 2 = 5 \\
a_5 &= a_4 + a_3 = 5 + 3 = 8 \\
a_6 &= a_5 + a_4 = 8 + 5 = 13 \\
a_7 &= a_6 + a_5 = 13 + 8 = 21 \\
a_8 &= a_7 + a_6 = 21 + 13 = 34 \\
a_9 &= a_8 + a_7 = 21 + 34 = 55
\end{aligned}$$

(Aside: Do you recognize this famous sequence?) □

Problem §8.1 - 26:

(a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

(b) What are the initial conditions for the recurrence relation in part (a)?

(c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

Solution. Let a_n denote the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

(a) We can use the strategy suggested in the hint in the textbook and break the set of ways to tile a $2 \times n$ checkerboard into two cases, for $n \geq 2$:

1. The right-most domino is positioned vertically, so the n^{th} column is filled and we need to tile the remaining $2 \times (n-1)$ board with dominoes. We can do so in a_{n-1} ways.
2. The right-most domino is positioned horizontally. In this case, there must be a second horizontal domino beneath it so the n^{th} and $(n-1)^{\text{st}}$ columsn are filled and we need to tile the remaining $2 \times (n-2)$ board using dominoes. We can do so in a_{n-2} ways.

This suggests the recurrence relation $a_n = a_{n-1} + a_{n-2}$ (the Fibonacci recurrence!).

(b) We can then determine initial conditions. There is exactly one way to tile a 1×2 board (with one vertical domino) and two ways to tile a 2×2 board (two vertical dominoes or two horizontal dominoes). Hence, $a_1 = 1$ and $a_2 = 2$.

(c) This part is asking us to compute a_{17}. We could do this by iteration, but we can also cheat by recognizing this sequence as the Fibonacci sequence:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 144, 233, 377, 610, 9871597, 2584, \cdots$$

and simply read off that $a_{17} = 2584$.

□

Problem §8.2 - 2: Classify each recurrence relation by stating (i) whether it is linear or nonlinear, (ii) whether it is homogeneous or nonhomogeneous, (iii) its order, and (iv) if it has constant coefficients.

(a) $a_n = 3a_{n-2}$

(b) $a_n = 3$

(c) $a_n = a_{n-1}^2$

(d) $a_n = a_{n-1} + 2a_{n-3}$

(e) $a_n = a_{n-1}/n$

(f) $a_n = a_{n-1} + a_{n-2} + n + 3$

(g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

Solution. Based on the definitions given in lecture and in the textbook, we can classify these recurrence relations as:

(a) Linear, homogeneous, constant coefficients, order 2

(b) Linear, nonhomogeneous, constant coefficients, order 0

(c) Not linear, homogeneous, constant coefficients, order 1

(d) Linear, homogeneous, constant coefficients, order 3

(e) Linear, homogeneous, non-constant coefficients, order 1

(f) Linear, nonhomogeneous, constant coefficients, order 2

(g) Linear, homogeneous, constant coefficients, order 7

□

Problem §8.2 - 4(a,d,e): Solve each recurrence relation along with the given initial conditions.

(a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

(d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$

(e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$

Solution. For each part, we follow the same general procedure: we find the characteristic equation and characteristic roots, write down the general solution, use the initial conditions to solve for the arbitrary constants in the general solution, and then write down the unique solution to the recurrence relation and given initial conditions.

Notice that each recurrence relation is a linear homogeneous recurrence relation with constant coefficients.

(a) Assume $a_n = r^n$ for some constant r. Substituting into the recurrence relation, we can compute the characteristic equation as

$$\begin{aligned} r^n &= r^{n-1} + 6r^{n-2} \\ r^2 &= r + 6 \\ 0 &= r^2 - r - 6 \\ 0 &= (r-3)(r+2) \end{aligned}$$

From this, we observe that there are two characteristic roots $r_1 = 3$ and $r_2 = -2$, each of which have multiplicity one. Hence, the general solution has form

$$a_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot 3^n.$$

Substituting for the initial conditions, we obtain the system of equations

$$\begin{aligned} &(n=0) \qquad \alpha_1 + \alpha_2 = a_0 = 3 \\ &(n=1) \qquad -2\alpha_1 + 3\alpha_2 = a_1 = 6 \end{aligned}$$

Solving for α_1 and α_2, we find that $\alpha_1 = 3/5$ and $\alpha_2 = 12/5$. Hence,

$$a_n = \frac{3}{5} \cdot (-2)^n + \frac{12}{5} \cdot 3^n.$$

(d) Assume $a_n = r^n$ for some constant r. We can then compute the characteristic equation as

$$\begin{aligned} r^n &= 2r^{n-1} - r^{n-2} \\ r^2 &= 2r - 1 \\ 0 &= r^2 - 2r + 1 \\ 0 &= (r-1)^2 \end{aligned}$$

Observe that there is a single characteristic root $r_0 = -1$ with multiplicity two. Hence, the general solution has form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot n \cdot 1^n = \alpha_1 + \alpha_2 n.$$

Substituting for the initial conditions, we obtain the system of equations

$$\begin{aligned} &(n=0) \qquad \alpha_1 = a_0 = 4 \\ &(n=1) \qquad \alpha_1 + \alpha_2 = a_1 = 1 \end{aligned}$$

from which we find $\alpha_1 = 4$ and $\alpha_2 = 1 - \alpha_1 = 1 - 4 = -3$. Hence,

$$a_n = 4 - 3n$$

(e) Assume that $a_n = r^n$ for some constant r. We can then compute the characteristic equation as

$$\begin{aligned} r^n &= r^{n-2} \\ r^2 &= 1 \\ 0 &= r^2 - 1 \\ 0 &= (r+1)(r-1) \end{aligned}$$

Observe that there are two distinct characteristic roots, $r_1 = -1$ and $r_2 = 1$, each of which have multiplicity one. Hence, the general solution has the form

$$a_n = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 1^n = \alpha_1 \cdot (-1)^n + \alpha_2.$$

Substituting for the initial conditions, we obtain the system of equations

$$\begin{aligned} (n=0) \qquad & \alpha_1 + \alpha_2 = a_0 = 5 \\ (n=1) \qquad & -\alpha_1 + \alpha_2 = a_1 = -1 \end{aligned}$$

Adding these equations together, we find that

$$2\alpha_2 = 5 + (-1) = 4$$

and therefore $\alpha_2 = 2$ and $\alpha_1 = 5 - \alpha_2 = 5 - 2 = 3$. Hence,

$$a_n = 3 \cdot (-1)^n + 2.$$

□

Problem §8.2 - 28:

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(b) Find all solutions of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Solution. (a) The associated linear homogeneous recurrence relation is $a_n = 2a_{n-1}$. Substituting for $a_n = r^n$, we can find the characteristic root(s):

$$\begin{aligned} a_n &= 2a_{n-1} \\ r^n &= 2r^{n-1} \\ r &= 2 \end{aligned}$$

Observe that there is a single characteristic root, with multiplicity one. Hence, we have $a_n^{(h)} = \alpha \cdot 2^n$, where α is a constant.

Next, we can find the particular solution. Observe that we could rewrite the given function as $F(n) = 2 \cdot n^2 \cdot 1^n$. Because 1 is not a characteristic root, we know from Theorem 6 that the particular solution has form

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) \cdot 1^n = p_2 n^2 + p_1 n + p_0$$

Now, we need to solve for the constants p_2, p_1, and p_0. We can do so by first plugging the particular solution into the nonhomogeneous recurrence relation:

$$p_2 n^2 + p_1 n + p_0 = 2\left(p_2(n-1)^2 + p_1(n-1) + p_0\right) + 2n^2$$

Simplifying and grouping like terms, we find:

$$-(p_2 + 2)n^2 + (4p_2 - p_1)n + (-2p_2 + 2p_1 - p_0) = 0$$

Equating like coefficients on the LHS and RHS, we obtain the system of equations

$$\begin{aligned} p_2 + 2 &= 0 \\ 4p_2 - p_1 &= 0 \\ -2p_2 + 2p_1 - p_0 &= 0 \end{aligned}$$

From the first equation, we see that $p_2 = -2$. Plugging this into the second equation, we find that $p_1 = 4p_2 = -8$. Finally, from the third equation we observe that $p_0 = 2p_1 - 2p_2 = -12$. Hence, our particular solution is

$$a_n^{(p)} = -(2n^2 + 8n + 12)$$

and the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha \cdot 2^n - (2n^2 + 8n + 12)$$

(Note: We can't solve for α unless we're given an initial condition.)

(b) Now that we've been given the initial condition $a_1 = 4$, we can solve for the unknown constant α in the general solution that we found in (a). To do so, we first substitute for the given initial condition

$$a_1 = 4 = \alpha \cdot 2^1 - (2 \cdot 1^2 + 8 \cdot 1 + 12) = 2\alpha - 22$$

and then solve for α, to find $\alpha = 13$. Hence, the solution to the given nonhomogeneous linear recurrence relation with the given initial condition is

$$a_n = 13 \cdot 2^n - (2n^2 + 8n + 12).$$

□

Problem §8.5 - 5: Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

(a) the sets are pairwise disjoint.

(b) there are 50 common elements in each pair of sets and no elements in all three sets.

(c) there are 50 common elements in each pair of sets and 25 elements in all three sets.

(d) the sets are equal.

Solution. For any triple of sets A_1, A_2, A_3, we know by the principle of inclusion-exclusion that

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

In each part, we simply determine the cardinality of each of the sets on the RHS and then use the above formula to compute $|A_1 \cup A_2 \cup A_3|$.

(a) If the sets are pairwise disjoint, then we know that $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 0$ and our formula reduces to

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 100 + 100 + 100 = 300$$

(b) Now, we're told that $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_2 \cap A_3| = 50$ and $|A_1 \cap A_2 \cap A_3| = 0$, hence

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &= 3 \cdot 100 - 3 \cdot 50 \\ &= 150\end{aligned}$$

(c) Now, we're told that $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_2 \cap A_3| = 50$ and $|A_1 \cap A_2 \cap A_3| = 25$, hence

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\ &= 3 \cdot 100 - 3 \cdot 50 + 25 \\ &= 175\end{aligned}$$

(d) We can answer this question in two ways. On one hand, we could simply observe that if $A_1 = A_2 = A_3$, then $|A_1 \cup A_2 \cup A_3| = |A_1| = |A_2| = |A_3| = 100$. On the other hand, suppose that we wanted to use the PIE formula. Then we could observe that $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 100$ and therefore

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\ &= 3 \cdot 100 - 3 \cdot 100 + 100 \\ &= 100\end{aligned}$$

□

Problem §8.5 - 10: Find the number of positive integers not exceeding 100 that are not divisible by 5 or 7.

Solution. Let E_1 be the set of positive integers not exceeding 100 that are divisible by 5 and E_2 be the set of positive integers not exceeding 100 that are divisible by 7. We wish to find $|\overline{E_1 \cup E_2}|$. To do so, we'll make use of the fact that $|\overline{E_1 \cup E_2}| = 100 - |E_1 \cup E_2|$. By the principle of inclusion-exclusion, we know that

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|,$$

hence

$$|\overline{E_1 \cup E_2}| = 100 - |E_1| - |E_2| + |E_1 \cap E_2|$$

Observe that $|E_1| = \lfloor 100/5 \rfloor = 20$, $|E_2| = \lfloor 100/7 \rfloor = 14$, and $|E_1 \cap E_2| = \lfloor 100/(5 \cdot 7) \rfloor = 2$. Substituting for these values, we find

$$|\overline{E_1 \cup E_2}| = 100 - |E_1| - |E_2| + |E_1 \cap E_2| = 100 - 20 - 14 + 2 = 68$$

□

Problem §8.5 - 14: How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat*, or *bird*?

Solution. Let E_1, E_2, and E_3 be the sets of permutations containing, respectively, the strings *fish*, *rat*, and *bird*. The problem is then asking us to find $|\overline{E_1 \cup E_2 \cup E_3}|$. Again, we'll use the fact that

$$\begin{aligned} |\overline{E_1 \cup E_2 \cup E_3}| &= \left(\begin{array}{c} \text{total number of permutations of the} \\ \text{26 letters of the English alphabet} \end{array} \right) - |E_1 \cup E_2 \cup E_3| \\ &= 26! - |E_1 \cup E_2 \cup E_3| \end{aligned}$$

From the principle of inclusion-exclusion, we know that

$$|E_1 \cup E_2 \cup E_3| = (|E_1| + |E_2| + |E_3|) - (|E_1 \cap E_2| + |E_2 \cap E_3| + |E_1 \cap E_3|) + |E_1 \cap E_2 \cap E_3|.$$

To find $|E_1|$, let the four letters in *fish* to be a glued superletter or block and count the number of permutations of that superletter and the remaining twenty-two ordinary letters. Hence, $|E_1| = 23!$. By the same argument, $|E_2| = 24!$ and $|E_3| = 23!$. To find $|E_1 \cap E_2|$, we can form two superletters by gluing together *fish* and *rat* and then permute these superletters and the remaining 20 normal letters. Hence, $|E_1 \cap E_2| = 21!$. Because *fish* and *bird* share the letter i and it's therefore not possible for a permutation to contain both strings, $|E_1 \cap E_3| = 0$. Similarly, $|E_2 \cap E_3| = 0$ and $|E_1 \cap E_2 \cap E_3| = 0$. Therefore,

$$|E_1 \cup E_2 \cup E_3| = 2 \cdot 23! + 24! - 21!$$

and

$$|\overline{E_1 \cup E_2 \cup E_3}| = 26! - 2 \cdot 23! - 24! + 21!$$

□

Problem §8.5 - 20: How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1,000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?

Solution. This is a simple application of the principle of inclusion-exclusion to compute the cardinality of the union of five sets E_1, E_2, E_3, E_4, and E_5. We're told that $|E_i| = 10,000$, $|E_i \cap E_j| = 1,000$, $|E_i \cap E_j \cap E_k| = 100$, $|E_i \cap E_j \cap E_k \cap E_\ell| = 10$, and $|E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 = 1$.

Using the principle of inclusion-exclusion, we can compute

$$\begin{aligned}|E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5| &= \left(\sum_{1\le i\le 5} |E_i|\right) - \left(\sum_{1\le i<j\le 5} |E_i \cap E_j|\right) + \left(\sum_{1\le i<j<j\le 5} |E_i \cap E_j \cap E_k|\right) \\ &\quad - \left(\sum_{1\le i<j<k<\ell\le 5} |E_i \cap E_j \cap E_k \cap E_\ell|\right) + |E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5| \\ &= 5 \cdot 10,000 - \binom{5}{2} \cdot 1,000 + \binom{5}{3} \cdot 100 - \binom{5}{4} \cdot 10 + 1 \\ &= 40,951\end{aligned}$$

$\square$