Problem §7.1 - 27(a): Find the probability of selecting exactly one of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding 40.

Solution. The sample space S is the set of all possible ways to choose six integers (without repetition) from the set $\{1, 2, \ldots, 40\}$. By definition, there are $\binom{40}{6}$ ways to do so.

The event E is the set of ways to choose six integers from $\{1, 2, \ldots, 40\}$ such that *exactly one* of those integers is correct. To count the number of ways to select exactly one correct integer, we can first choose the correct integer - there are exactly six ways to do so. Then, we need to fill the remaining five spots in our selection with incorrect integers - i.e., we need to choose five integers from the set of thirty-four incorrect integers, which we can do in $\binom{34}{5}$ ways. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{6 \cdot \binom{34}{5}}{\binom{40}{6}} = \frac{139,128}{319,865} \approx 43.5\%$$

□

Problem §7.1 - 36: Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled? Show your work.

Solution. In this problem, we want to compare the probabilities of two events:

$$E_1 = \text{rolling a total of 8 when rolling 2 dice}$$
$$E_2 = \text{rolling a total of 8 when rolling 3 dice}$$

It's important to note, however, that the events E_1 and E_2 live in different sample spaces because we're rolling different numbers of dice. Event E_1 lives in the sample space S_1, i.e. the set of all possible outcomes when rolling two dice. Event E_2 lives in the sample space S_2, i.e. the set of all possible outcomes when rolling three dice. Since the dice are six sided, by definition $|S_1| = 6^2$ and $|S_2| = 6^3$.

To compare these probabilities, the best tactic is actually going to be to just calculate the probabilities and then directly compare. To do so, we need to determine $|E_1|$ and $|E_2|$. The easiest way to do so is actually via brute force. We can represent the outcome of rolling two dice as a pair (a, b). We then see that there are exactly five ways to get a total of 8: $(6, 2), (2, 6), (5, 3), (3, 5)$, and $(4, 4)$. So $|E_1| = 5$.

Similarly, we can represent the outcome of rolling three dice as a triple (a, b, c). By running through each of the possible values of a, we can enumerate the triples in E_2 as:

$$(6,1,1),\ (5,2,1),\ (5,1,2),\ (4,3,1),\ (4,1,3),\ (4,2,2),\ (3,4,1),\ (3,1,4),\ (3,3,2)\ ,(3,2,3),(2,5,1),$$
$$(2,1,5),\ (2,4,2),\ (2,2,4),\ (2,3,3),\ (1,6,1),\ (1,1,6),\ (1,5,2),\ (1,2,5),\ (1,4,3),\ (1,3,4)$$

Hence, $|E_2| = 21$.

We can then calculate

$$p(E_1) = \frac{|E_1|}{|S_1|} = \frac{5}{6^2} \approx 13.9\%$$
$$p(E_2) = \frac{|E_2|}{|S_2|} = \frac{21}{6^3} \approx 9.7\%$$

Comparing $p(E_1)$ and $p(E_2)$, we see that the probability of rolling an 8 with two dice is higher than the probability of rolling an 8 with three dice. □

Problem §7.2 - 8(a,c,d): What is the probability of these events when we randomly select a permutation of $\{1, 2, \ldots, n\}$ where $n \geq 4$?

(a) 1 precedes 2.

(c) 1 immediately precedes 2.

(d) n precedes 1 and $n - 1$ precedes 2.

Solution. In these problems, we want to compute the probability of randomly selecting a permutation of $\{1, 2, \ldots, n\}$, where $n \geq 4$, with certain properties. The sample space S is the same for all of these questions: the set of all possible permutations of $\{1, 2, \ldots, n\}$. By definition, $|S| = n!$.

In each part, we'll compute the desired probability by computing the cardinality of the event E that a permutation has the specified properties.

(a) Observe that in a permutation of $\{1, 2, \ldots, n\}$, we must have either 1 preceding 2 or 2 preceding 1. Since these are equally likely, we can quickly see that the probability that 1 precedes 2 is simply 1/2.

(Note that this reasoning is much faster than any brute force argument)

(c) In contrast to (a), now 1 must *immediately* precede 2. In Chapter 6, we counted permutations of sets of letters where certain letters had to appear as a block by thinking about those blocks of letters as "superletters" and then counting arrangements of the superletters and ordinary letters. The same tactic applies here; we can think about instead permuting the set $\{12, 3, 4, \ldots, n\}$. Because the 1 and 2 have merged into a single supernumber, we are now counting permutations of $n - 1$ objects. By definition, there are $(n - 1)!$ such permutations. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

(d) We can make use of the same logic as in (a). Half of the permutations have n preceding 1. Of those permutations, half have $n-1$ preceding 2. Hence, exactly one fourth of the permutations have n preceding 1 *and* $n - 1$ preceding 2, so the probability is 1/4.

□

Problem §7.2 - 18:

(a) What is the probability that two people chosen at random were born on the same day of the week?

(b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?

(c) How many people chosen at random are needed to make the probability greater than 1/2 that there are at least two people born on the same day of the week?

Solution. As per the instructions in the textbook, we assume that the births are independent and that being born on each day of the week is equally likely - i.e., the probability of being born on a given weekday is 1/7.

(a) Here, the sample space S is the set of all possible pairs of weekdays and the event E is the set of all outcomes where the two people are born on the same day. Since there are seven days of the week, clearly $|S| = 7^2 = 49$. To count the number of outcomes in E, we simply need to count the number of ways to choose the shared day of the week. Clearly, $|E| = 7$. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{7}{49} = \frac{1}{7}$$

(b) Now, the sample space S is the set of all possible choices of n weekdays (with repetition allowed). From Chapter 6, we know that $|S| = 7^n$.

The event E is the set of outcomes where at least two of the n people are born on the same day of the week. Note that because there are only seven days of the week, when $n \geq 8$ the pigeonhole principle guarantees that at least two people must be born on the same day of the week (here, the people are the pigeons and the days of the week are the pigeonholes). Note also that trivially $p(E) = 0$ if $n = 1$. Hence, $p(E) = 1$ for $n \geq 8$ and we need only consider $2 \leq n < 8$.

For $2 \leq n < 8$, it's easier for us to compute the probability of the *complementary event* $\overline{E}$ that each person is born on a distinct day of the week. We can do so as

$$p(\overline{E}) = \frac{7}{7} \cdot \frac{6}{7} \cdots \frac{8-n}{7}$$

and therefore

$$p(E) = 1 - p(\overline{E}) = 1 - \cdot \frac{6}{7} \cdots \frac{8-n}{7}$$

(Note that this formula actually holds for $n \geq 8$ as well, since then $8 - n = 0$!)

1. By brute force, we can use the formula we developed for $p(E)$ in part (b) to determine when this probability exceeds $1/2$. Plugging in various values for n, we observe that

$$\begin{aligned}
\text{for } n = 2: &\quad p(E) = 1 - \frac{6}{7} = \frac{1}{7} < \frac{1}{2} \\
\text{for } n = 3: &\quad p(E) = 1 - \frac{6}{7} \cdot \frac{5}{7} = \frac{19}{49} < \frac{1}{2} \\
\text{for } n = 4: &\quad p(E) = 1 - \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} = \frac{223}{343} > \frac{1}{2}
\end{aligned}$$

From these computations, we see that the probability will exceed $1/2$ when $n \geq 4$.

□

Problem §7.2 - 24: What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Solution. Let the sample space S be the set of all possible outcomes when flipping a fair coin five times. Then let the event E be the set of all outcomes where the first flip is tails and the event F be the set of all outcomes where there are exactly four heads.

By definition, we know the conditional probability that there are exactly four heads given that the first flip came up tails is given by

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

In event F, the first coin flip is predetermined and the other four coin flips can be either heads or tails. Hence, $|F| = 2^4$. The intersection $E \cap F$ consists of the single outcome where the first coin flip is tails and the other four coin flips are all heads (this is the only way that four flips can be heads). Hence, $|E \cap F| = 1$. We can then compute

$$p(E \cap F) = \frac{p(E \cap F)}{p(F)} = \frac{|E \cap F|}{|S|} \cdot \frac{|S|}{|F|} = \frac{|E \cap F|}{|F|} = \frac{1}{16}$$

(Notice that we don't need to bother computing $|S|$, since it just cancels out!) □

Problem §7.2 - 30: Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if

(a) a 0 bit and a 1 bit are equally likely.

(b) the probability that a bit is 1 is 0.6.

(c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.

Solution. We can think about this as a series of n independent Bernoulli trials, one for each bit in the binary string. Here, "success" is generating a 1 and "failure" is generating a 0. In each part, we're told the probability of success, p, in each trial and we want to compute the probability that all n trials are successes. From lecture (or Theorem 2 in the textbook), we know that the probability of exactly n successes is

$$C(10, 10)p^{10}(1 - p)^{10-10} = 1 \cdot p^{10} \cdot (1 - p)^0 = p^{10}.$$

(a) If 0 and 1 are equally likely, then $p = 1/2$ and the probability of generating a string with only "1"s is $(1/2)^{10}$.

(b) Now, $p = 0.6$ and the probability of generating a string with only "1"s is $(0.6)^{10}$.

(c) Now, the probability that the ith bit is a 1 is $1/2^i$. So to compute the probability that we generate a binary string of all "1"s, we need to multiply the probabilities that each individual bit is a "1":

$$\frac{1}{2} \cdot \frac{1}{2^2} \cdots \frac{1}{2^{10}} = \frac{1}{2^{1+2+\cdots+10}} = \frac{1}{2^55} \approx 2.8 \times 10^{-17},$$

which is essentially zero.

□

Problem §7.2 - 34: Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

(a) the probability of no successes.

(b) the probability of at least one success.

(c) the probability of at most one success.

(d) the probability of at least two successes.

Solution. To answer this problem, we can use the binomial distribution. Recall that the probability of exactly k successes in a set of n independent Bernoulli trials is

$$b(k; n, p) = C(n, k)p^k(1 - p)^{n-k}.$$

(a) If we have no successes, $k = 0$. Plugging this in, we compute

$$b(0; n, p) = C(n, 0)p^0(1 - p)^{n-0} = 1 \cdot p^0(1 - p)^n = (1 - p)^n.$$

(b) In order to compute the probability of at least one success, it's easier to compute the probability of the *complementary event* that there are *no* successes. Since we already did that in (a), we can use our existing work to compute

$$p\left(\geq 1 \text{ successes}\right) = 1 - b(0; n, p) = 1 - (1 - p)^n.$$

(c) We can break the set of outcomes where there is at most one success into two cases: no successes and exactly one success. We can therefore compute the probability of at most one success as

$$\begin{aligned} p(\leq 1 \text{ success}) &= p(\text{no successes}) + p(1 \text{ success}) \\ &= b(0; n, p) + b(1; n, p) \\ &= (1-p)^n + C(n,1)p^1(1-p)^{n-1} \\ &= (1-p)^n + np(1-p)^{n-1} \end{aligned}$$

(Note that this formula only makes sense if $n > 0$ - if $n = 0$, then we're performing no trials and so the probability is clearly 1)

(d) Again, it's easier to compute this probability by computing the probability of the complementary event. Since the complementary event is that there is at most one success, we've actually already computed it in (c). Hence, we can compute the desired probability as

$$\begin{aligned} p(\geq 2 \text{ successes}) &= 1 - p(\leq 1 \text{ successes}) \\ &= 1 - \left[(1-p)^n + np(1-p)^{n-1}\right] \end{aligned}$$

□

Problem §7.2 - 36: Use mathematical induction to prove that if $E_1, E_2, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then

$$p\left(\cup_{i=1}^n E_i\right) = \sum_{i=1}^n p(E_i).$$

Solution. Let $E_1, E_2, \ldots E_n$ be a sequence of n pairwise disjoint events in a sample space S. We will prove by induction that

$$p\left(\cup_{i=1}^n E_i\right) = \sum_{i=1}^n p(E_i).$$

Before doing so, we note that the claim is trivially true for $n = 1$ because it reduces to the claim that $p(E_i) = p(E_i)$. We will use $n = 2$ as our base case rather than $n = 1$, as that's the case required in our inductive argument.

Base Case: When $n = 2$, we know that E_1 and E_2 being pairwise disjoint means that $E_1 \cap E_2$, so

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = p(E_1) + p(E_2).$$

Inductive Hypothesis: Assume that if $E_1, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, then

$$p\left(\cup_{i=1}^n E_i\right) = \sum_{i=1}^n p(E_i).$$

Inductive Step: Now, we wish to show that the identity holds for any sequence of $n+1$ pairwise disjoint events in a sample space S. Let $E_1, \ldots, E_{n+1}$ be such a sequence of pairwise disjoint events. Observe that if we let $F = E_1 \cup \cdots \cup E_n$, then we can write

$$p(\cup_{i=1}^{n+1} E_n) = p(F \cup E_{n+1}).$$

Observe that F is also an event in the sample space S. To verify that F and E_{n+1} are disjoint events, observe that $(E_1 \cup \cdots \cup E_n) \cap E_{n+1} = (E_1 \cap E_{n+1}) \cup \cdots \cup (E_n \cap E_{n+1})$ because, as we verified earlier in the course, we have distributive laws for set intersection and union. Because $E_1, \ldots, E_{n+1}$

are by assumption pairwise disjoint, we know that $E_i \cap E_{n+1} = \emptyset$ for all $1 \leq i \leq n$ and therefore $(E_1 \cup \cdots \cup E_n) \cap E_{n+1} = \emptyset$ and we conclude that F and E_{n+1} are disjoint events. This means that we can apply the base case, and write

$$\begin{aligned} p(F \cup E_{n+1}) &= p(F) + p(E_{n+1}) \\ &= p(E_1 \cup \cdots \cup E_n) + p(E_{n+1}). \end{aligned}$$

We can then apply the inductive hypothesis to $p(E_1 \cup \cdots \cup E_n)$, to obtain

$$\begin{aligned} p(E_1 \cup \cdots \cup E_n) + p(E_{n+1}) &= \left(\sum_{i=1}^{n} p(E_i) \right) + p(E_{n+1}) \\ &= \sum_{i=1}^{n+1} p(E_i) \end{aligned}$$

Conclusion: Because we've explicitly verified that the identity holds for $n = 1$ and $n = 2$ and then shown that it holding for any sequence of pairwise disjoint events $E_1, \ldots, E_n$ implies that it also holds for any sequence of pairwise disjoint events $E_1, \ldots, E_{n+1}$, we have shown by mathematical induction that

$$p\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} p(E_i)$$

for all $n \in \mathbb{Z}_{\geq 0}$, as desired. □

Problem §7.3 - 2: Suppose that E and F are events in a sample space and $p(E) = 2/3$, $p(F) = 3/4$, and $p(F \mid E) = 5/8$. Find $p(E \mid F)$.

Solution. From the definition of conditional probability, we know that

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)},$$

so to compute $P(E \mid F)$ we need to know $p(F)$ and $p(E \cap F)$. We're given $p(F) = 3/4$. Since we're not given $p(E \cap F)$ directly, we'll have to be a little clever in computing it from the given information. By definition, we know that

$$p(F \mid E) = \frac{p(E \cap F)}{p(E)}.$$

Since the only unknown in that equation is $p(E \cap F)$, we can solve for it and plug in the given values to compute

$$p(E \cap F) = p(E)p(F \mid E) = \frac{2}{3} \cdot \frac{5}{8} = \frac{5}{12}.$$

We can then use this to compute

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{5}{12} \cdot \frac{4}{3} = \frac{5}{9}.$$

□

Problem §7.3 - 8: Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% without the disease test positive.

(a) What is the probability that someone who tests positive has the genetic disease?

(b) What is the probability that someone who tests negative does not have the disease?

Solution. First, let's recast the information from this problem in the language of discrete probability. We know that we are likely to use Bayes' Theorem, so we can begin by computing some of the probabilities that we'll need.

Let D be the event that a randomly chosen person has the rare genetic disease. In the problem statement, we're told $p(D) = |D|/|S| = 1/10,000 = 0.0001$ so $p(\overline{D}) = 1 - p(D) = 0.9999$. Then, let P be the event that a randomly chosen person tests positive for the rare disease. In the problem statement, we're told:

$$p(P \mid D) = 0.999 \qquad \text{(a "true positive")},$$
$$p(P \mid \overline{D}) = 0.0002 \qquad \text{(a "false positive")}.$$

From these probabilities, we can compute

$$p(\overline{P} \mid D) = 1 - p(P \mid D) = 1 - 0.999 = 0.001 \qquad \text{(a "false negative")},$$
$$p(\overline{P} \mid \overline{D}) = 1 - p(P \mid \overline{D}) = 1 - 0.0002 = 0.9998 \qquad \text{(a "true negative")}.$$

We're now ready to address specific parts of the question.

(a) This is asking us for the probability $p(D \mid P)$ (the probability of having the disease given a positive test). Using Bayes' Theorem and the probabilities computed above, we can calculate

$$\begin{aligned} p(D \mid P) &= \frac{p(P \mid D)p(D)}{p(P \mid D)p(D) + p(P \mid \overline{D})p(\overline{D})} \\ &= \frac{0.999 \cdot 0.0001}{0.999 \cdot 0.0001 + 0.0002 \cdot 0.9999} \\ &\approx 0.3331 \end{aligned}$$

(b) Now, we're being asked for $p(\overline{D} \mid \overline{P})$ (the probability of not having the disease given a negative test). Again, we can use Bayes' Theorem and the previously computed probabilities to find

$$\begin{aligned} p(\overline{D} \mid \overline{P}) &= \frac{p(\overline{P} \mid \overline{D})p(\overline{D})}{p(\overline{P} \mid \overline{D})p(\overline{D}) + p(\overline{P} \mid D)p(D)} \\ &= \frac{0.9998 \cdot 0.0002}{0.9998 \cdot 0.0002 + 0.001 \cdot 0.0001} \\ &\approx 0.9995, \end{aligned}$$

which we observe is extremely close to 1!

□