Problem §6.3 - 16: How many subsets with an odd number of elements does a set with 10 elements have?

Solution. Recall that the number of r element subsets of an n element set is given by $C(n,r) = \binom{n}{r}$. A subset with an odd number of elements could contain 1, 3, 5, 7, or 9 elements. Hence, using the sum rule we can count the number of subsets with an odd number of elements by computing

$$\begin{aligned} C(10,1) + C(10,3) + C(10,5) + C(10,7) + C(10,9) &= \binom{10}{1} + \binom{10}{3} + \binom{10}{5} + \binom{10}{7} + \binom{10}{9} \\ &= 10 + 120 + 252 + 120 + 10 \\ &= 512 \end{aligned}$$

□

Problem §6.3 - 20: How many bit strings of length 10 have

(a) exactly three "0"s?

(b) more "0"s than "1"s?

(c) at least seven "1"s?

(d) at least three "1"s?

Solution. One key insight for this problem is that each entry in a binary string is either a "0" or a "1". So once you've determined the positions of all "0"s (or equivalently, all "1"s), the rest of the string is determined.

(a) We can choose locations for the three "0"s in $C(10,3) = \binom{10}{3} = \frac{10!}{3!7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2} = 120$ ways. Once these locations are chosen, the rest of the string is determined. Hence, there are 120 such strings.

(b) In order for there to be more "0"s than "1"s, the string must contain strictly less than 5 "1"s. One brute force method is to break this into cases based on the number of "1"s, count the number of ways to choose positions for the "1"s in each case, and then apply the sum rule. Doing so, we compute

$$\begin{aligned} C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) &= \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} \\ &= 1 + 10 + 45 + 120 + 210 \\ &= 386 \end{aligned}$$

There's a more elegant way to think about this, though. Consider the set of strings where there are *not* exactly five "0"s. In half of these cases, there are more "0"s than "1"s (in the other half, there are more "1"s than "0"s). Hence, we can use the idea of counting the complement to compute

$$\begin{aligned} \left(\begin{array}{c}\text{Number of strings with} \\ \text{more "0"s than "1"s}\end{array}\right) &= \frac{\left(\begin{array}{c}\text{Total number of length 10} \\ \text{binary strings}\end{array}\right) - \left(\begin{array}{c}\text{Number of strings with} \\ \text{exactly five "0"s}\end{array}\right)}{2} \\ &= \frac{2^{10} - C(10,5)}{2} \\ &= 386 \end{aligned}$$

Happily, this matches our brute force computation!

(c) Once again, we can employ the sum rule to count the number of binary strings with 7, 8, 9, or 10 "1"s. Doing so, we find that there are

$$\begin{aligned} C(10,7)+C(10,8)+C(10,9)+C(10,10) &= \binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} \\ &= 120+45+10+1 \\ &= 176 \end{aligned}$$

such strings.

(d) Here, it's faster to count the *complement* - i.e., to count the number of strings that contain *less than* three "1"s. Doing so, we compute that

$$\begin{aligned} \left(\begin{array}{c}\text{Number of strings with} \\ \text{more at least three "1"s}\end{array}\right) &= \left(\begin{array}{c}\text{Total number of length 10} \\ \text{binary strings}\end{array}\right) - \left(\begin{array}{c}\text{Number of strings with} \\ \text{less than three "1"s}\end{array}\right) \\ &= 2^{10} - (C(10,0)+C(10,1)+C(10,2)) \\ &= 2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} \\ &= 1024-1-10-45 \\ &= 968 \end{aligned}$$

□

Problem §6.3 - 24: How many ways are there for ten women and six men to stand in a line so that no two men stand next to each other?

Solution. We can break this into two tasks: (1) choosing positions for the women to stand, relative to each other and then (2) choosing positions to "insert" the men into this line of women.

Counting the number of ways to do task (1) is relatively straightforward - this is just asking the number of ways to permute a set of ten women, which is by definition $P(10,10)$. When thinking about task (2), we need to be careful to satisfy the condition that no two men stand next to each other. If we represent each woman with a $*$ and each available position for a man with a _, we can draw the following rough diagram of the available positions for men to stand:

$$\_*\_*\_*\_*\_*\_*\_*\_*\_*\_*\_$$

There's an open position to the left of the first woman, nine positions between each pair of women, and an open position to the right of the tenth woman - a total of eleven possible positions. Since there are six men, we can choose positions for the men in $P(11,6)$ ways.

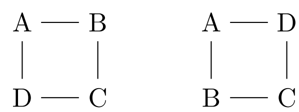
By the product rule, we can then compute that the number of ways to position ten women and six men in a line so that no two men are standing next to each other is:

$$\begin{aligned} \left(\begin{array}{c}\text{number of ways to} \\ \text{position the group}\end{array}\right) &= \left(\begin{array}{c}\text{number of ways to} \\ \text{position the women}\end{array}\right) \cdot \left(\begin{array}{c}\text{number of ways to} \\ \text{"insert" the men}\end{array}\right) \\ &= P(10,10)\cdot P(11,6) \\ &= 10! \cdot \frac{11!}{5!} \\ &= 1,207,084,032,000. \end{aligned}$$

□

Problem §6.3 - 42: Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

Solution. This is a slight variation on a type of problem that we've seen before, in Homework 4. In Homework 4 (specifically, #44 from §6.1), we counted the number of ways to seat a subset of a group of people at a round table. In that example, we considered two seatings to be the same if everyone had the same immediate left and right neighbors. The difference in this question is that now we're not distinguishing between left and right neighbors. For example, in §6.1 : #44 we would have distinguished between the seatings



but now we want to count these as the same seating.

When $r = 1$, the problem is trivial because there's only one way to seat a single person at a table. So this is just the number of ways to choose a subset of size one, i.e. $\binom{n}{1} = n$. When $r = 2$, there is again only one way to seat a pair of people at a round table. So this is just the number of ways to choose a subset of size two, i.e. $\binom{n}{2}$. For the remainder of the problem, we will therefore assume that $r \geq 3$.

We can count the number of ways to seat r of n people around a circular table by by designating the "head" of the table and then seating the rest of the people going clockwise. There are $P(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ ways to do so. Because the table is round, however, this overcounts by a factor of r (it distinguishes between a seating and the same seating where each person has rotated one seat clockwise, etc). Hence, there are $\frac{n!}{r(n-r)!}$ distinct such seatings.

But in the context of *this* problem, we're still overcounting! Now that we don't care which side the neighbors are sitting on, we could reverse the order of people around the table (from clockwise to counterclockwise or vice versa) and get the same seating. Hence, we need to divide by 2.

So we conclude that for $r \geq 3$, there are

$$\frac{n!}{2r(n-r)!}$$

such distinct ways to seat r of n people around a circular table. □

Problem §6.4 - 10: Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.

Solution. From the binomial theorem, we know that

$$\left(x + \frac{1}{x}\right)^{100} = \sum_{j=0}^{100} \binom{100}{j} x^{100-j} \left(\frac{1}{x}\right)^j.$$

Doing a little bit of algebra, we can rewrite this as

$$\sum_{j=0}^{100} \binom{100}{j} x^{100-j} \left(\frac{1}{x}\right)^j = \sum_{j=0}^{100} \binom{100}{j} x^{100-j} x^{-j} = \sum_{j=0}^{100} \binom{100}{j} x^{100-2j}.$$

We would like to find an expression for the coefficient of x^k in this expansion. First, we can observe that x^k has a non-zero coefficient exactly when $k = 100-2j$ for some $j = 0, \ldots, 100$. We can observe that this is when $k = -100, -98, \ldots, -2, 0, 2, \ldots, 98, 100$ - i.e., when k is an even integer between -100 and 100. Solving this expression for j, we obtain $j = (100 - k)/2$.

Hence, we conclude that

$$\text{The coefficient of } x^k = \begin{cases} \binom{100}{(100-k)/2} & \text{if } k = -100, -98, \ldots, 98, 100 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem §6.4 - 12: The row of Pascal's triangle containing the binomia coefficients $\binom{10}{k}$, for $0 \le k \le 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

Solution. Recall that Pascal's identity states that for positive integers n, k with $n \ge k$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

In a practical sense, this means that we can compute the entries on row $n+1$ of Pascal's triangle by adding adjacent numbers in the previous row (i.e., the 'left and right diagonal' entries for each entry in row $n+1$), starting and ending with a 1. Doing so, we compute the next row of Pascal's triangle as

1 11 55 165 330 462 462 330 165 55 11 1

□

Problem §6.4 - 22: Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are non-negative integers with $r \le n$ and $k \le r$,

(a) using a combinatorial argument.

(b) using an argument based on the formula for the number of r-combinations of a set with n elements.

Solution. (a) Suppose that we have a set $S = \{1, \ldots, n\}$ and we wish to choose a subset $A \subseteq S$ with k elements and another subset $B \subseteq S$ with $r - k$ elements, such that $A \cap B = \emptyset$ (i.e., A and B are disjoint subsets of S). Observe that together, A and B contain a total of r elements (each of which is either in A or in B, but *not* in both).

On one hand, we could start by choosing the r elements of S that will be in either A or B. There are clearly $\binom{n}{r}$ ways to do so. We could then choose which of these r elements will be in set A, in $\binom{r}{k}$ ways. All of the remaining $r - k$ elements must be in set B. By the product rule, this gives us $\binom{n}{r}\binom{r}{k}$ ways to select the disjoint subsets A and B. This is the expression on the LHS.

On the other hand, we could instead start by choosing k elements of S to place in subset A. There are $\binom{n}{k}$ ways to do so. Once we've chosen the elements of A, we can then choose the $r - k$ elements of B from the remaining $n - k$ elements of S in $\binom{n-k}{r-k}$ ways. By the product rule, this gives us $\binom{n}{k}\binom{n-k}{r-k}$ ways to choose the disjoint subsets A and B. This is the expression on the RHS.

Because both strategies were counting the same thing, we conclude that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k},$$

as desired.

(The assumptions that $r \le n$ and $k \le n$ are neatly handled by the fact that there are zero ways to choose a subset larger than the original set!)

(b) This part is just asking us to use the definition of a binomial coefficient and our algebraic skills to verify the identity. Using the definition of a binomial coefficient and multiplying by a clever form of 1, we observe that

$$\begin{aligned}\binom{n}{r}\binom{r}{k} &= \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!}\\ &= \frac{n!}{k!(n-r)!(r-k)!}\\ &= \frac{n!(n-k)!}{k!(r-k)!(n-k)!}\\ &= \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!}\\ &= \binom{n}{k}\binom{n-k}{r-k},\end{aligned}$$

as desired.

□

Problem §6.4 - 27(a): Prove the **hockeystick identity**

$$\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers, using a combinatorial argument.

Solution. We prove this identity by counting the number of binary strings of length $n+r+1$ that contain exactly r "0"s and $n+1$ "1"s.

On one hand, this is clearly $\binom{n+r+1}{r}$. We can just count the number of ways to choose which positions in the string have a "0".

Counting in a way that yields the formula on the LHS, however, takes a little bit more thought. The fact that we have a summation suggests that we probably want to break the set of binary strings of length $n+r+1$ with exactly r "0"s and $n+1$ "1"s into cases based on some condition. We'll do so based on the position of the last "1" that appears in the string. Denote this position by $\ell+1$. Because there are exactly $n+1$ "1"s in the string, we know that the *smallest* ℓ can be is n. This corresponds to the case where the first $n+1$ entries are "1" and the remaining entries are all "0". On the other extreme, ℓ can get as large as $n+r$. This corresponds to the case where the last "1" appears as the last entry of the string.

For each value of ℓ, we can count the number of binary strings with the last "1" in position $\ell+1$ by counting the number of ways to choose the positions of the other n "1"s. Because all of the other "1"s must appear before entry $\ell+1$, there are $\binom{\ell}{n}$ ways to choose the positions of the "1"s. Equivalently, there are $\binom{\ell}{n-\ell}$ ways to choose which of the first ℓ entries will contain "0"s. Although it's not immediately obvious, it turns out to be more convenient for us to express it this way. Using the sum rule, this gives us

$$\sum_{\ell=n}^{n+r}\binom{\ell}{\ell-n}.$$

If we compare this to the summation that we want, we can notice that if we make the change of variable $k=\ell-n$, then we can rewrite our summation as

$$\sum_{k=0}^{r}\binom{n+k}{k}.$$

Thus, we get the identity

$$\binom{n+r+1}{r} = \sum_{k=0}^{r} \binom{n+k}{k},$$

as desired. □

Problem §6.5 - 10(a,c,d): A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

(a) a dozen croissants?

(c) two dozen croissants with at least two of each kind?

(d) two dozen croissants with no more than two broccoli croissants?

Solution. (a) This is just asking for the number of ways to choose 12 croissants from a set of 6 flavors, with repetition allowed. There are

$$\binom{6+12-1}{12} = \binom{17}{12} = 6188$$

ways to do so.

(b) We want to choose a total of 24 croissants, with at least 2 of each kind. The requirement that we have at least 2 of each of the 6 flavors means that 12 of the croissants are already determined. We then want to count the number of ways to choose the remaining 12 croissants - which we already did in (a)! So there are still $\binom{17}{12} = 6188$ ways to do so.

(d) Here, it's easier to count the *complement* - i.e., the number of ways to choose at least three broccoli croissants. Assuming that 3 of our 24 croissants are broccoli, we then need to choose flavors for the remaining 21 croissants. We can do so in

$$\binom{6+21-1}{21} = \binom{26}{21} = 65,780$$

ways. If there had been no restrictions, we could have chosen 24 croissants in

$$\binom{6+24-1}{24} = \binom{29}{24} = 118,755$$

ways. Hence, we can compute the number of ways to choose two dozen croissants with no more than two broccoli croissants as

$$\begin{aligned}\left(\begin{array}{c}\text{ways to choose 24 croissants}\\ \text{with no more than 2 broccoli}\end{array}\right) &= \left(\begin{array}{c}\text{ways to choose}\\ \text{24 croissants}\end{array}\right) - \left(\begin{array}{c}\text{ways to choose 24 croissants}\\ \text{with at least 3 broccoli}\end{array}\right)\\ &= 118,755 - 65,780\\ &= 52,975\end{aligned}$$

□

Problem §6.5 - 20: How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are non-negative integers?

Solution. Like the hint in the textbook suggests, we can introduce a fourth variable x_4 and instead ask how many solutions there are to the equation

$$x_1 + x_2 + x_3 + x_4 = 11$$

where all $x_i \geq 0$. Each solution to this equation corresponds to a solution to the original inequality - we simply ignore x_4! Now, we know from many previous examples that there are

$$\binom{4+11-1}{11} = \binom{14}{11} = 364$$

solutions. □

Problem §6.5 - 26: How many positive integers less than $1,000,000$ have exactly one digit equal to 9 and have a sum of digits equal to 13?

Solution. Observe that numbers less than $1,000,000$ have at most six digits. Let x_i be the i-th digit of the number, for $1 \leq i \leq 6$. Then this is asking for the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$$

where each $x_i \geq 0$ and exactly one is equal to 9.

In order to count the number of solutions, we can first choose which digit is 9. There are 6 ways to do so. Without loss of generality, let's assume that we chose $x_6 = 9$. Then, we're really asking for the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 4$$

Because we know that exactly one digit of our number is 9, we now have the additional restriction that $0 \leq x_i \leq 8$ for $1 \leq i \leq 5$ (this prevents us from overcounting by counting any number that has 9 as more than one digit). We really don't have to think about this restriction, though, because the summation is only equal to 4 so it's not possible to have any other x_i be 9.

From much previous work, we know that the number of solutions to this equation is

$$\binom{5+4-1}{4} = 70.$$

□

Problem §6.5 - 46: A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

Solution. The textbook gave us a nice hint about how to do this problem using the stars and bars model. We'll represent the chosen books with bars and the unchosen books with stars, so we're arranging five bars and seven stars.

Observe that there are six positions where the stars can be placed (between each pair of bars, before the first bar, or after the last bar):

-|-|-|-|-|-

Because we can't choose adjacent books, we need to place at least one star in each of the positions that appears between a pair of bars. There are four such positions. We can then choose how to place the remaining $7 - 4 = 3$ stars in

$$\binom{6+3-1}{3} = \binom{8}{3} = 56$$

ways. □

In problems §7.1 : $10 - 20$ (even), the sample space S is the set of all possible five-card hands from a standard deck. By definition of an r-combination, we know that $|S| = \binom{52}{5}$.

Problem §7.1 - 10: What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?

Solution. We want to compute the probability of the event E that our five-card hand contains the two of diamonds and the three of spades. Because these two cards are fixed, we simply need to count the number of ways to choose the remaining three cards in the hand from the remaining fifty cards. There are by definition $\binom{50}{3}$ ways to do so. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{\binom{50}{3}}{\binom{52}{5}} = \frac{50!}{3! \cdot 47!} \cdot \frac{5! \cdot 47!}{52!} = \frac{5 \cdot 4}{52 \cdot 51} = \frac{5}{663} \approx 0.75\%$$

□

Problem §7.1 - 12: What is the probability that a five-card poker hand contains exactly one ace?

Solution. We want to compute the probability of the event E that our five-card hand contains exactly one ace. Because there are four suits, there are 4 ways to choose the ace. After doing so, we need to choose the other four cards in the hand from the remaining set of forty-eight cards (we can't choose a second ace, so we need to exclude all four aces) in $\binom{48}{4}$ ways. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{4 \cdot \binom{48}{4}}{\binom{52}{5}} = 4 \cdot \frac{48!}{4! \cdot 44!} \cdot \frac{5! \cdot 47!}{52!} = \frac{4 \cdot 5 \cdot 47 \cdot 46 \cdot 45}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{3243}{10829} \approx 29.9\%$$

□

Problem §7.1 - 14: What is the probability that a five-card poker hand contains cards of five different kinds?

Solution. We want to compute the probability of the event E that our five-card hand contains cards of five different kinds. To count the outcomes in E, we observe that there are $\binom{13}{5}$ ways to choose the five kinds of cards in the hand (since there thirteen kinds of cards in a standard deck). Once we've determined the kinds, we can then choose the suit for each card in four ways. Hence, $|E| = 4^5 \cdot \binom{13}{5}$ and therefore

$$p(E) = \frac{|E|}{|S|} = \frac{4^5 \cdot \binom{13}{4}}{\binom{52}{5}} = \frac{704}{2499} \approx 51\%$$

□

Problem §7.1 - 16: What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?

Solution. We want to compute the probability of the event E that our hand contains five cards of the *same* suit. To count the outcomes in E, we observe that there are $\binom{4}{1} = 4$ ways to choose the suit of the cards and then $\binom{13}{5}$ ways to choose the cards from that suit. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{33}{16660} \approx 0.2\%$$

□