Problem §5.2: 8: Suppose that a store offers gift certificates in denominations of 25 and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

Solution. First, we need to determine which total amounts can be formed using gift certificates in denominations of 25 and 40. Notice that $\$25 = 5(\$5)$ and $\$40 = 8(\$5)$, so every possible dollar amount has the form $\$5n$. Then observe that

$$\begin{aligned}
\$25 &= 1(\$25) & \$105 &= 2(\$40) + 1(\$25)\\
\$40 &= 1(\$40) & \$115 &= 1(\$40) + 3(\$25)\\
\$50 &= 2(\$25) & \$120 &= 3(\$40)\\
\$65 &= 1(\$40) + 1(\$25) & \$125 &= 5(\$25)\\
\$75 &= 3(\$25) & \$130 &= 2(\$40) + 2(\$25)\\
\$80 &= 2(\$40) & \$140 &= 1(\$40) + 4(\$25)\\
\$90 &= 1(\$40) + 2(\$25) & \$145 &= 3(\$40) + 1(\$25)\\
\$100 &= 4(\$25) & \$150 &= 5(\$25)\\
\$125 &= 5(\$25) & \$155 &= 2(\$40) + 3(\$25)\\
& & \$160 &= 4(\$40)
\end{aligned}$$

Because we've tested all possible combinations of the gift certificates, we know that there are no gaps in this list. Observe that once $n \geq 28$, we can form $\$5n$ out of $25 and $40 gift certificates.

Let $P(n)$ be the statement that we can form $\$5n$ out of the $25 and $40 gift certificates. We will prove that $P(n)$ holds for all $n \geq 28$ by strong induction.

Base Case: Above, we already explicitly verified that $P(n)$ is true for $n = 28, 29, 30, 31,$ and 32.

Inductive Hypothesis: Assume that $P(j)$ is true for all j with $28 \leq j \leq k$, where k is some fixed integer greater than or equal to 32.

Inductive Step: We want to show that $P(k+1)$ is true, i.e. that we can form $\$5(k+1)$ using $25 and $40 gift certificates. Observe that

$$\$5(k+1) = \$5k + \$5 = \$5k + \$5 - \$25 + \$25 = \$5k - \$20 + \$25 = \$5(k-4) + \$25.$$

Because $k \geq 32$, we know that $k - 4 \geq 28$ and therefore, by the inductive hypothesis, we can form $\$5(k-4)$ out of $25 and $40 gift certificates. Adding a single $25 gift certificate gives us $\$5(k+1)$, as desired.

Conclusion: Because we verified the base cases $P(n)$ for $n = 28, 29, 30, 31,$ and 32 and we showed that assuming $P(j)$ for all $28 \leq j \leq k$ implies $P(k+1)$, we've shown by strong induction that we can form $\$5n$ from $25 and $40 gift cards for all $n \geq 28$, as desired. □

Problem §5.2: 10: Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along on a vertical or horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Solution. We will prove via strong induction that it takes $n - 1$ breaks to separate a rectangular chocolate bar with n squares into n individual squares. Denote this statement as $P(n)$.

Base Case: If $n = 1$ (i.e., the bar contains only one square), then this is trivially true because it takes zero breaks to reduce the bar to individual squares.

Inductive Hypothesis: Assume that for all $1 \leq j \leq k$, for some fixed k, it requires $j - 1$ breaks to reduce a chocolate bar with j squares to individual squares.

Inductive Step: Consider a chocolate bar with $k+1$ squares. If we want to break this bar into individual squares, the process requires beginning with one initial break. This produces two new bars. Depending on how we broke the bar, this produces two new bars - one with $i+1$ squares and another with $k-i$ squares, for some $0 \leq i \leq k-1$. Because both $i+1$ and $k-i$ are less than k, we can apply the inductive hypothesis. By the inductive hypothesis, reducing a chocolate bar with $i+1$ squares to individual squares requires $(i+1)-1 = i$ breaks and reducing a chocolate bar with $k-i$ squares to individual squares requires $k-i-1$ breaks. Therefore, the total number of breaks required is

$$i + (k-i-1) + 1 = k + (i-i) + (1-1) = k,$$

as claimed.

Conclusion: Because we verified the base case, $P(0)$, and showed that assuming that $P(0), \dots, P(k)$ implies that $P(k+1)$ is also true, we have shown by strong induction that it takes exactly $n-1$ breaks to reduce a chocolate bar with n squares to individual squares, as desired. □

Problem §6.1: 8: How many different three-letter initials with none of the letters repeated can people have?

Solution. Because each letter must be unique, there are 26 choices for the first initial, 25 choices for the second, and then 24 choices for the third. Hence, by the product rule there are a total of

$$26 \cdot 25 \cdot 24 = 15,600$$

different three-letter initials with no letters repeated. □

Problem §6.1: 14: How many bit strings of length n, where n is a positive integer, start and end with 1s?

Solution. First, we need to deal with the special case $n = 1$. If $n = 1$, then there is exactly one such string: 1. If $n \geq 2$, then we want to count strings of the form

$$1\_\_ \cdots \_\_1$$

Because we have two choices (0 or 1) to fill each of the $n-2$ interior positions of the string, there are 2^{n-2} such strings. □

Problem §6.1: 16: How many strings are there of four lowercase letters that have the letter x in them?

Solution. We can count the number of strings of four lowercase letters that contain the letter x by instead counting the *complement* of that set. Doing so, we find

$$\begin{aligned}\left(\begin{array}{c}\text{Number of length 4 strings}\\ \text{that contain x}\end{array}\right) &= \left(\begin{array}{c}\text{total number of}\\ \text{length 4 strings}\end{array}\right) - \left(\begin{array}{c}\text{number of length 4 strings}\\ \text{that don't contain x}\end{array}\right)\\ &= 26^4 - 25^4\\ &= 66,351\end{aligned}$$

□

Problem §6.1: 26: How many strings of four decimal digits

(a) do not contain the same digit twice?

(b) end with an even digit?

(c) have exactly three digits that are 9s?

Solution. (a) If the digits can't be repeated, then there are 10 ways to choose the first digit, 9 ways to choose the second digit, 8 ways to choose the third digit, and 7 ways to choose the fourth digit. Hence, by the product rule there are $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$ such strings.

(b) Now, we allow repetition but require that the last digit be even, i.e. be from the set $\{2, 4, 6, 8\}$. This means that there are 10 ways to choose each of the first three digits and 5 ways to choose the last digit. Hence, by the product rule there are $10^3 \cdot 5 = 5,000$ such strings.

(c) To construct a string of four decimal digits with exactly three digits that are 9s, we need to complete two tasks: choose the position of the digit that isn't a 9, and then choose the digit that will go in that space from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. There are 4 ways to choose the position and 9 choices of the digit to place in that position. Hence, by the product rule there are $4 \cdot 9 = 36$ such strings.

□

Problem §6.1: 30: How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

Solution. We can answer this question by using both the sum and product rules. First, we can count the number of license plates with three uppercase English letters followed by three digits. There are 26 choices for each letter and 10 choices for each digit, so by the product rule there are $26^3 \cdot 10^3$ such license places. Then, we can count the number of license plates with four uppercase English letters followed by two digits. Again, the product rule tells us that there are $26^4 \cdot 10^2$ such license plates. Because any valid license plate has to be formed in one of these two ways, we know by the sum rule that there are

$$26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 63,273,600$$

possible valid license plates formed in one of these two ways. □

Problem §6.1: 36: How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?

Solution. For each element of the domain, we have two choices of function value. Hence, by the product rule there are 2^n such functions. □

Problem §6.1: 37: How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Solution. (a) If $n > 2$, then it's not possible to have a one-to-one function from the set $\{1, 2, \ldots, n\}$ to $\{0, 1\}$. If $n = 1$, then there are two such functions - either $f(1) = 0$ or $f(1) = 1$. If $n = 2$, then there are still two such functions because once the image of 1 is determined, the image of 2 is forced to be the other element.

(b) If $n = 1$, then the function is totally determined and there is only one such function. If $n > 1$, then we still need to determine where the function maps the elements $2, \ldots, n - 1$. Observe that there are exactly $n - 2$ elements in this subset. Each of those elements can be mapped to either 0 or 1. Hence, by the product rule, there are 2^{n-2} such functions.

(c) If $n = 1$, then there are no positive integers less than n and therefore there are no such functions. Hence, we only need to think about the $n > 1$ case. In order to define such a function, we need to: (1) decide which number between 1 and $n-1$ will be mapped to 1, and then (2) determine the function value of n. For (1), we have $n-1$ choices. For (2), we have two choices because n is free to be mapped to either 0 or 1. Hence, by the product rule there are $2 \cdot (n-1)$ such functions.

$\square$

Problem §6.1: 40: How many subsets of a set with 100 elements have more than one element?

Solution. Again, we can use the idea of counting the *complement* of a set. Observe that

$$\begin{pmatrix} \text{number of subsets} \\ \text{with } \geq 1 \text{ elements} \end{pmatrix} = \begin{pmatrix} \text{total number} \\ \text{of subsets} \end{pmatrix} - \begin{pmatrix} \text{number of subsets} \\ \text{with } \leq 1 \text{ elements} \end{pmatrix}.$$

In total, there are 2^{100} subsets. We can see this by observing that if we construct a subset, we have two choices for each element of the set: we can either include or not include that element. If we think about which of these subsets contain no more than one element, there are 101: the empty set and 100 sets consisting of one element. Hence, the number of subsets with more than one element is $2^{100} - 101$. $\square$

Problem §6.1: 44: How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

Solution. If we were just counting the number of ways to arrangement four people from a set of ten people, there would be $10 \cdot 9 \cdot 8 \cdot 7$ such arrangements. This overcounts the number of ways to seat the four people at a round table, however, because we can rotate a seating arrangement around the table in four ways and still have the same seating arrangement. Hence, by the division rule there are

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4} = \frac{5040}{4} = 1260$$

distinct seating arrangements of four people from a group of ten around a circular table. $\square$

Problem §6.2: 2: Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Solution. Here, the students are the "pigeons" and the letters of the English alphabet are the "pigeonholes". Because there are 30 students and 26 letters of the alphabet, the pigeonhole principle guarantees that at least two students must have last names beginning with the same letter. $\square$

Problem §6.2: 6: Let d be a positive integer. Show that among any group of $d+1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Solution. Let S be some set of $d+1$ (not necessarily consecutive) integers. Then here the "pigeons" are the elements of S and the "pigeonholes" are the possible remainders when divided by d. Observe that when an integer is divided by d, there are d possible remainders: $0, 1, \ldots, d-1$. Because there are more integers than remainders, the pigeonhole principle guarantees that at least two of these integers must have the same remainder. $\square$

Problem §6.2: 8: Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

Solution. This is a simple application of the pigeonhole principle. Here, the "pigeons" are the elements of the domain, S, and the "pigeonholes" are the elements of the codomain, T. Because $|S| > |T|$, the pigeonhole principle guarantees that at least two elements of the domain must be mapped to the same element of the codomain. In other words, there must exist some distinct elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$. □

Problem §6.2: 14:

(a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

(b) Is the conclusion in (a) true if six integers are selected rather than seven?

Solution. (a) We can partition the first ten positive integers into subsets of integers that sum to eleven: $\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, and $\{5, 6\}$. Doing so allows us to frame this problem as an application of the pigeonhole principle. Here, the "pigeons" are the integers $1, \ldots, 10$ and the "pigeonholes" are the five subsets of elements that sum to eleven. If we select seven integers from the set $\{1, \ldots, 10\}$, then the pigeonhole principle guarantees that two integers must be in the same subset and therefore sum to 11. If we "forget" these two integers, then we still have five integers and four remaining subsets. So again, the pigeonhole principle guarantees that at least two of the elements must be in the same subset and sum to 11. This gives us two pairs of integers that sum to 11, as desired.

(b) No, the conclusion is no longer true. For example, we could select the set $\{1, 2, 3, 4, 5, 6\}$ which contains only one pair with the desired sum: $5 + 6 = 11$.

□