Problem §3.2: 2(a,b,e,f): Determine whether each of these functions is $O(x^2)$:

(a) $f(x) = 17x + 11$.

(b) $f(x) = x^2 + 1000$.

(e) $f(x) = 2^x$.

(f) $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$.

Solution. (a) **Yes**, $f(x)$ is $O(x^2)$ because

$$|17x + 11| \le |17x + x| = |18x| \le |18x^2|$$

for all $x > 11$. The witnesses are $C = 18$ and $k = 11$.

(b) **Yes**, $f(x)$ is $O(x^2)$ because

$$|x^2 + 1000| \le |x^2 + x^2| = 2x^2$$

for all $x > \sqrt{1000}$. The witnesses are $C = 2$ and $k = \sqrt{1000}$.

(c) **No**, $f(x)$ is not $O(x^2)$. If it were, then we would have $|2^x| < C|x^2|$ for some constant C, but $2^x > x^3$ for all $x \ge 10$. So for large x, $|2^x/x^2| \ge |x^3/x^2| = |x|$ which is certainly not bounded by a constant.

(f) **Yes**, $f(x)$ is $O(x^2)$ because

$$|\lfloor x \rfloor \lceil x \rceil| \le |x(x+1)| \le x(2x) = 2x^2$$

for all $x > 1$. The witnesses are $C = 2$ and $k = 1$.

□

Problem §3.2: 8: Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

(a) $f(x) = 2x^2 + x^3 \log x$.

(b) $f(x) = 3x^5 + (\log x)^4$.

(c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$.

(d) $f(x) = (x^3 + 5 \log x)/(x^4 + 1)$.

Solution. For each function, we essentially want to identify its fastest growing term and find an $O(x^n)$ bound for that term.

(a) The fastest growing term is $x^3 \log x$. This term is not $O(x^3)$ because the $\log x$ factor grows without bound as x grows. Because $\log x$ grows more slowly than x, this suggests that it may be $O(x^4)$. To verify this, we observe that

$$|2x^2 + x^3 \log x| \le |2x^4 + x^4| = 3|x^4|$$

for all $x > 1$. As such, $f(x)$ is $O(x^4)$ with witnesses $C = 3$ and $k = 1$.

(b) The fastest growing term is $3x^5$. To see that $f(x)$ is $O(x^5)$, observe that

$$|3x^5 + (\log x)^4| \le |3x^5 + x^5| = 4|x^5|$$

for all $x > 1$. As such, $f(x)$ is $O(x^5)$ with witnesses $C = 4$ and $k = 1$.

(c) Informally, we can observe that if we took the limit of $f(x)$ as $x \to \infty$, this function would approach 1. This suggests that the function is $O(1)$. To verify this, observe that

$$\left|\frac{x^4+x^2+1}{x^4+1}\right| \leq \left|\frac{x^4+x^4+x^4}{x^4+1}\right| \leq \left|\frac{x^4+x^4+x^4}{x^4}\right| = \left|\frac{3x^4}{x^4}\right| = 3\cdot|1|,$$

so $f(x)$ is $O(1)$ with witnesses $C = 3$ and $k = 1$.

(d) Again, we can informally observe when x is very large, $f(x) \approx 1/x$. This suggests that the function is $O(1/x)$. To verify this, observe that

$$\begin{aligned}\left|\frac{x^3+5\log x}{x^4+1}\right| &\leq \left|\frac{x^3+5x^3}{x^4+1}\right| \qquad \text{for } x > 0\\ &\leq \left|\frac{6x^3}{x^4}\right|\\ &= 6\left|\frac{1}{x}\right|\end{aligned}$$

so $f(x)$ is $O(1/x)$ with witnesses $C = 6$ and $k = 0$.

□

Problem §3.2: 17: Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$. Show that $f(x)$ is $O(h(x))$.

Solution. Assume that $f(x)$ is $O(g(x))$, so by definition there exist constants C_1 and k_1 such that $|f(x)| \leq C_1|g(x)|$ for $x > k_1$. Assume also that $g(x)$ is $O(h(x))$, so by definition there exist constants C_2 and k_2 such that $|g(x)| \leq C_2|h(x)|$ for $x > k_2$. Then observe that

$$|f(x)| \leq C_1|g(x)| \leq C_1C_2|h(x)|$$

when $x > \max(k_1, k_2)$. Hence, $f(x)$ is by definition $O(h(x))$ with witnesses $C = C_1C_2$ and $k = \max(k_1, k_2)$. □

Problem §3.2: 26: Give a big-O estimate for each of these functions. For the function g in your estimate $f(x)$ is $O(g(x))$, use a simple function g of the smallest order.

(a) $f(x) = (n^3 + n^2\log n)(\log n + 1) + (17\log n + 19)(n^3 + 2)$.

(b) $f(x) = (2^n + n^2)(n^3 + 3^n)$.

(c) $f(x) = (n^n + n2^n + 5^n)(n! + 5^n)$.

Solution. As in Problem 8 from §3.2, we want to identify the fastest growing term. Discarding any constant multiple, this term gives a smallest order big-O estimate for the function.

(a) We look at each term independently. The first term is the product $(n^3 + n^2\log n)(\log n + 1)$. The fastest growing term of the $n^3 + n^2\log n$ factor is n^3 and the fastest growing term of the $\log n + 1$ factor is $\log n$. So this term "grows like" $n^3\log n$. The second term is the product $(17\log n + 19)(n^3 + 2)$. By similar reasoning, this term "grows like" $n^3\log n$. This means the overall funtion is $O(n^3\log n + n^3\log n) = O(2n^3\log n)$. Because constant coefficients aren't important when thinking about big-O estimates, this is equivalent to being $O(n^3\log n)$.

(b) Again, we want to identify the fastest growing term in each factor. The first factor $2^n + n^2$ has fastest growing term 2^n. The second factor $n^3 + 3^n$ has fastest growing term 3^n. As such, $f(x)$ is $O(2^n \cdot 3^n) = O(6^n)$.

(c) The fastest growing term in the first factor, $n^n + n2^n + 5^n$, is n^n. The fastest growing factor in the second term, $n! + 5^n$, is $n!$. As such, $f(x)$ is $O(n^n n!)$.

□

Problem §3.2: 28(a,b,c,d): Determine whether each of the following functions is $\Omega(x)$ and whether it is $\Theta(x)$.

(a) $f(x) = 10$.

(b) $f(x) = 3x + 7$.

(c) $f(x) = x^2 + x + 1$.

(d) $f(x) = 5 \log x$.

Solution. One strategy for finding a big-Theta estimate for for $f(x)$ is, as in previous problems, to look at the fastest growing term.

(a) **No**, this function is not $\Theta(x)$. We know that a function $f(x)$ is $\Theta(x)$ if and only if $f(x)$ is $O(x)$ and x is $O(f(x))$. Clearly $f(x) = 10$ is $O(x)$ because $10 < |x|$ for all $x > 10$. Observe however that x is not $O(10)$ because there are no constants C, k for which $|x| < 10$ for all $x > k$. As such, $f(x) = 10$ is not $\Theta(x)$.

(b) **Yes**, this function is $\Theta(x)$. Observe that $|3x + 7| \leq 4|x|$ for $x > 7$ and that $|3x + 7| \geq 3|x|$ for $x > 0$ (in fact, this is true for all x so we could have said $x > k$ for any choice of k). By definition, this means $f(x) = 3x + 7$ is $O(x)$ and $\Omega(x)$ and is therefore $\Theta(x)$.

(c) **No**, this function is not $\Theta(x)$. The leading term, x^2, grows more quickly than x and therefore $f(x)$ is not $O(x)$ or $\Theta(x)$. (It is, however, $\Omega(x)$!)

(d) **No**, this function is not $\Theta(x)$. The function $\log x$ grows more slowly than x, so $f(x)$ is not $\Omega(x)$ or $\Theta(x)$. (It is, however, $O(x)$!)

□

Problem Extra: Explain what it means for a function to be

(a) $O(1)$.

(b) $\Omega(1)$.

(c) $\Theta(1)$.

Solution. (a) By definition, a function $f(x)$ is $O(1)$ if there exist constants C, k such that $|f(x)| \leq C$ for all $x > k$. In other (more intuitive) words, a function $f(x)$ is $O(1)$ if its absolute value is bounded **above** for all $x > k$.

(b) By definition, a function $f(x)$ is $\Omega(1)$ if there exist constants C, k such that $|f(x)| > C$ for all $x > k$. In other (more intuitive) words, a function $f(x)$ is $\Omega(1)$ if its absolute value is bounded **below** for all $x > k$ so $f(x)$ "stays away" from zero for large enough x. For example, the function $1/x$ is not $\Omega(1)$ because $\lim_{x\to\infty} 1/x = 0$ but the function $x - 5$ is because, for example, $|x - 5| \geq 3$ for $x > 7$.

(c) By definition, a function $f(x)$ is $Theta(1)$ if there exist positive constants C_1, C_2, and k such that

$$C_1 \leq |f(x)| \leq C_2$$

for all $x > k$. This means that $|f(x)|$ is bounded between two positive constants. So for large x, $|f(x)|$ can't get "too large" or too close to zero.

□

Problem §5.1: 4: Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

(f) Explain why these steps show that this formula is true whenever n is a positive integer.

Solution. (a) $P(1)$ is the statement

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2.$$

(b) We can easily verify that both sides of $P(1)$ are equal to 1:

$$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1,$$
$$1^3 = 1.$$

(c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

We denote this statement as $P(k)$.

(d) In the inductive step, we want to show that $P(k)$ implies $P(k+1)$ for each $k \geq 1$. That is, we want to show that by assuming the inductive hypothesis from part (c) we can prove that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

(e) Beginning with the left hand side of $P(k+1)$, we can observe that

$$\begin{aligned}
\left(1^3 + 2^3 + \cdots + k^3\right) + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \qquad \text{(by the IHOP)} \\
&= (k+1)^2 \left(\frac{k^2}{4} + k + 1\right) \\
&= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4}\right) \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2,
\end{aligned}$$

as desired.

(f) Because we've shown that the statement holds for the base case, $n = 1$, and that $P(k)$ implies $P(k+1)$, we know by the principle of mathematical induction that the statement $P(n)$ holds for all positive integers n.

□

Problem §5.1: 6: Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Solution. We will use mathematical induction to show that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

for all positive integers n.

Base Case: When $n = 1$, we can verify that $1 \cdot 1! = 1 = 2! - 1$.

Inductive Step: Assume that $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$ for $k \in \mathbb{Z}_{>0}$. Observe that

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \qquad \text{(by the IHOP)} \\ &= (k+2) \cdot (k+1)! - 1. \end{aligned}$$

Conclusion: Because we've shown that the identity holds for $n = 1$ and that $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$ implies $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$ for all $k \in \mathbb{Z}_{>0}$, we conclude by the principle of mathematical induction that the claim holds for all $n \in \mathbb{Z}_{>0}$, as desired. □

Problem §5.1: 8: Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

Solution. We wish to show that for all $n \in \mathbb{N}$,

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = \frac{1 - (-7)^{n+1}}{4}.$$

Base Case: When $n = 0$, the left-hand side has only the single term 2 and the right-hand side simplifies to

$$\frac{1 - (-7)^{0+1}}{4} = \frac{1 - (-7)}{4} = \frac{8}{4} = 2.$$

Because both sides of the identity are equal to 2, $P(0)$ is true.

Inductive Step: Assume $P(k)$ is true, i.e. that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k = (1 - (-7)^{k+1})/4$. We can then observe that

$$\begin{aligned} 2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} &= \frac{1 - (-7)^{k+1}}{4} + 2(-7)^{k+1} \qquad \text{(by the IHOP)} \\ &= \frac{1 - (-7)^{k+1} + 8(-7)^{k+1}}{4} \\ &= \frac{1 + 7(-7)^{k+1}}{4} \\ &= \frac{1 - (-7)(-7)^{k+1}}{4} \\ &= \frac{1 - (-7)^{k+2}}{4}. \end{aligned}$$

Conclusion: Because we've shown that the identity holds for $n = 0$ and that $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{N}$, we have therefore shown by mathematical induction that

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = \frac{1 - (-7)^{n+1}}{4}$$

for all $n \in \mathbb{N}$, as desired. □

Problem §5.1: 20: Prove that $3^n < n!$ if n is an integer greater than 6.

Solution. We wish to show that $3^n < n!$ for all integers $n > 6$.

Base Case: The smallest integer greater than 6 is $n = 7$, so that's the appropriate base case. To verify that the statement holds for $n = 7$, observe that $3^7 = 2187$, $7! = 5040$, and so $3^7 < 7!$.

Inductive Step: Assume that $3^k < k!$ for some integer $k > 6$. Then observe that

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &< (k+1) \cdot 3^k \qquad \text{(because } k > 6\text{, so } k+1 > 7 > 3\text{)} \\ &< (k+1) \cdot k! \qquad \text{(by the IHOP)} \\ &= (k+1)! \end{aligned}$$

Conclusion: Because we verified the base case $3^7 < 7!$ and showed that $3^k < k!$ implies $3^{k+1} < (k+1)!$ for all $k > 6$, we have shown by the principle of mathematical induction that $3^n < n!$ for all integers $n > 6$, as desired. □

Problem §5.1: 34: Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

Solution. We wish to show that 6 divides $n^3 - n$ for all nonnegative integers n.

Base Case: The smallest nonnegative integer is $n = 0$. When $n = 0$, the statement trivially holds because $6 \mid 0$ (This is standard notation that means "6 divides 0").

Inductive Step: Suppose that $6 \mid (k^3 - k)$ for some nonnegative integer k. We wish to show that 6 also divides $(k+1)^3 - (k+1)$. To see this, observe that

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k) \\ &= (k^3 - k) + 3k(k+1) \end{aligned}$$

We know that $6 \mid (k^3 - k)$ by the inductive hypothesis. Observe that the second term, $3k(k+1)$, is clearly divisible by 3. It must also be divisible by 2, because either k or $k+1$ must be even. As such, the second term must also be divisible by 6 and therefore 6 must divide the entire expression, i.e. $6 \mid ((k+1)^3 - (k+1))$.

Conclusion: Because we showed that $6 \mid (0^3 - 0)$ and that having $6 \mid (k^3 - k)$ implies $6 \mid ((k+1)^3 - (k+1))$ for all integers $k \geq 0$, we have therefore shown by the principle of mathematical induction that $6 \mid (n^3 - n)$ for all nonnegative integers n, as desired. □

Problem §5.1: 49: What is wrong with this "proof" that all horses are the same color?

Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses: number these horses as $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

Solution. This "proof" has the same flaw as Example 3 on the "Errors in Inductive Proofs" worksheet. The problem is that the argument in the inductive step is not valid when $k = 1$. When $k = 1$, the inductive step tells us to divide a set of two horses into a set containing just the first horse and a set containing just the last horse. In this case, the statement "Because the set of the first k horses and the set of the last k horses overlap..." is nonsense - the set containing just the first horse and the set containing just the second horse are disjoint. □

Problem §5.1: 51: What is wrong with this "proof"?

"*Theorem*": For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

Solution. Again, the problem with this "proof" is in the inductive step. The inductive step applies the inductive hypothesis to $\max(x-1, y-1)$. However, this implicitly assumes that $x-1$ and $y-1$ are positive integers whenever x and y are positive integers. This is not always true - for $k = 1$, we could have, for example, $x = 1$ and $y = 2$. Then $x-1 = 0$ is not a positive integer and the inductive hypothesis doesn't apply. □