Problem §2.3: 2: Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

(a) $f(n) = \pm n$

(b) $f(n) = \sqrt{n^2 + 1}$

(c) $f(n) = \frac{1}{n^2 - 4}$

Solution. (a) **No**. This is not a function because $f(n)$ is not well-defined, i.e. it does not map each element of the domain to a *single* element of the codomain.

(b) **Yes**. For all $z \in \mathbb{Z}$, the image $f(z) = \sqrt{z^2 + 1}$ is well-defined and lies in the codomain, $\mathbb{R}$.

(c) **No**, because $f(z)$ is not defined for all $z \in \mathbb{Z}$. Observe that for both $z = 2$ and $z = -2$, $f(z)$ is undefined because it would involve division by zero. In order for $f(n)$ to be a function with domain $\mathbb{Z}$, it would need to be defined on all elements of $\mathbb{Z}$.

□

Problem §2.3: 12: Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

(a) $f(n) = n - 1$.

(b) $f(n) = n^2 + 1$.

(c) $f(n) = n^3$.

(d) $f(n) = \lceil n/2 \rceil$.

Solution. Recall that a function $f : A \to B$ is one-to-one if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for all $a_1, a_2 \in A$. To show that a function is not one-to-one, it is sufficient to find a single counterexample where $a_1 \neq a_2$ but $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$.

Yes, this function is one-to-one. For any $n_1, n_2 \in \mathbb{Z}$, observe that if $f(n_1) = f(n_2)$ then

$$f(n_1) = n_1 - 1 = n_2 - 1 = f(n_2),$$

which implies that $n_1 = n_2$.

(b) **No**, this function is not one-to-one. Observe that, for example,

$$f(-2) = (-2)^2 + 1 = 5 = 2^2 + 1 = f(2),$$

but $-2 \neq 2$.

(c) **Yes**, this function is one-to-one. For any $n_1, n_2 \in \mathbb{Z}$, observe that if $f(n_1) = f(n_2)$ then

$$f(n_1) = n_1^3 = n_2^3 = f(n_2),$$

which implies that $n_1 = n_2$ because all real numbers have a unique cube root.

(d) **No**, this function is not one-to-one. For example, observe that

$$f(3) = \lceil 3/2 \rceil = 2 = \lceil 4/2 \rceil = f(4),$$

but $3 \neq 4$.

□

Problem §2.3: 14(a,b,c,d): Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m, n) = 2m - n$.

(b) $f(m, n) = m^2 - n^2$.

(c) $f(m, n) = m + n + 1$.

(d) $f(m, n) = |m| - |n|$.

Solution. Recall that a function $f : A \to B$ is onto if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. To show that a function is not onto, it's sufficient to find a single $b \in B$ that is not the image of any element of the domain, A.

(a) **Yes**, this function is onto. Observe that any integer z in the codomain, $\mathbb{Z}$, is the image of $(0, -z)$:

$$f(0, -z) = 2(0) - (-z) = z.$$

(b) **No**, this function is not onto. For example, 2 is in its codomain but not its range. Observe that if

$$m^2 - n^2 = (m - n)(m + n) = 2,$$

then m and n must have the same parity, i.e. must both either be even or odd (if m and n had different parities, then both $m - n$ and $m + n$ would be odd, forcing their product, $m^2 - n^2$, to also be odd). If m and n have the same parity, then both $m - n$ and $m + n$ are even and therefore divisible by 2. Hence, their product is divisible by 4 and cannot be equal to 2.

(c) **Yes**, this function is onto. Observe that any integer z in the codomain $\mathbb{Z}$ is the image of $(0, z - 1)$:

$$f(0, z - 1) = 0 + (z - 1) + 1 = z.$$

(d) **Yes**, this function is onto. Observe that any positive integer z in the codomain $\mathbb{Z}$ is the image of $(z, 0)$, any negative integer z is the image of $(0, z)$, and 0 is the image of $(0, 0)$:

$$\begin{aligned} f(z, 0) &= |z| - |0| = |z| = z, && \text{for } z \in \mathbb{Z}_{\geq 0}, \\ f(0, z) &= |0| - |z| = -|z| = -(-z) = z, && \text{for } z \in \mathbb{Z}_{\leq 0}, \\ f(0, 0) &= |0| - |0| = 0. \end{aligned}$$

□

Problem §2.3: 20: Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

(a) one-to-one but not onto.

(b) onto but not one-to-one.

(c) both onto and one-to-one (but not the identity function).

(d) neither one-to-one nor onto.

Solution. (a) The function $f(n) = n + 1$ is one-to-one but not onto. To see that it's one-to-one, observe that for all $n_1, n_2 \in \mathbb{N}$, if $f(n_1) = f(n_2)$ then $n_1 + 1 = n_2 + 1$ which implies $n_1 = n_2$. It's not onto, however, because 0 is not the image of any natural number. To see this, observe that if we had $0 = f(n) = n + 1$, this would require that $n = -1$ and $-1 \notin \mathbb{N}$.

(b) The function $f(n) = \lceil n/2 \rceil$ is onto but not one-to-one. Observe that any element n of the codomain is the image of both $2n$ and $2n + 1$:

$$\begin{aligned} f(2n) &= \lceil (2n)/2 \rceil = \lceil n \rceil = n, \\ f(2n + 1) &= \lceil \frac{2n + 1}{2} \rceil = \lceil n + 1/2 \rceil = n. \end{aligned}$$

(c) Consider the piecewise function

$$f(n) = \begin{cases} n-1 & \text{n even} \\ n+1 & \text{n odd} \end{cases}$$

which "swaps" the even and odd natural numbers. For example, $f(1) = 2$ and $f(2) = 1$, $f(3) = 4$ and $f(4) = 3$, etc. This function is onto because each even n in the codomain is the image of $n-1$ and each odd n in the codomain is the image of $n+1$. It is also one-to-one. To see that it is one-to-one, observe that if $f(n_1) = f(n_2)$, then either $n_1 - 1 = n_2 - 1$ or $n_1 + 1 = n_2 + 1$, depending on parity. In either case, this implies $n_1 = n_2$.

(d) The function $f(n) = 0$ is clearly neither onto nor one-to-one because it maps every element of the domain to the same element of the codomain.

□

Problem §2.3: 22(a,b): Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = -3x + 4$.

(b) $f(x) = -3x^2 + 7$.

Solution. Recall that a bijection is a function that is both injective (one-to-one) and surjective (onto). So one strategy would be to determine if each function is both injective or surjective. To save ourselves some work, though, when we want to show that a function is a bijection we can use the fact that only bijections have inverses. Showing that an inverse function exists is, therefore, equivalent to showing that the function is a bijection.

(a) **Yes**, this function is a bijection. We claim that the inverse function of f is $f^{-1}(x) = (4-x)/3$. To verify this, observe that for $x \in \mathbb{R}$,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(-3x+4) = \frac{4-(-3x+4)}{3} = \frac{3x}{3} = x,$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{4-x}{3}\right) = -3\left(\frac{4-x}{3}\right) + 4 = -4 + x + 4 = x$$

(b) **No**, this function is not a bijection because it's not injective *or* surjective. To see that it's not injective, observe that, for example, $f(-1) = -3(-1)^2 + 7 = 4 = -3(1)^2 + 7 = f(1)$, but $-1 \neq 1$. To see that it's not surjective, observe that $x^2 \geq 0$ for all $x \in \mathbb{R}$. As such, the range of $f(x)$ is $(-\infty, 7]$, which is clearly not equal to the codomain $\mathbb{R}$.

□

Problem §2.3: 36: Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution. Because both f and g are functions from $\mathbb{R}$ to $\mathbb{R}$, the compositions $f \circ g$ and $g \circ f$ are well-defined. We can compute these compositions as:

$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5,$$
$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3.$$

Notice that $g \circ f \neq f \circ g$! □

Problem §2.3: 39: Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

Solution. One easy way to show that the given function f is invertible is to exhibit an inverse function. We claim that it has inverse function

$$\begin{aligned} f^{-1} &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \frac{x-b}{a}. \end{aligned}$$

To verify that this is the inverse function of f, we need to check that both $(f \circ f^{-1})$ and $(f^{-1} \circ f)$ are the identity function on $\mathbb{R}$. We can do so by computing

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) = f\left(\frac{x-b}{a}\right) = a\left(\frac{x-b}{a}\right) + b = x - b + b = x, \\ (f^{-1} \circ f)(x) &= f^{-1}(f(x)) = f^{-1}(ax+b) = \frac{(ax+b)-b}{a} = \frac{ax}{a} = x. \end{aligned}$$

□

Problem §2.3: 40(a): Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that $f(S \cup T) = f(S) \cup f(T)$.

Solution. We'll show that $f(S \cup T) = f(S) \cup f(T)$ by showing that each set is a subset of the other.

First, suppose that $b \in f(S \cup T)$. By definition, this means that $b = f(a)$ for some $a \in S \cup T$. By definition of union, either $a \in S$, $a \in T$, or both. If $a \in S$, then $f(a) \in f(S)$. If $a \in T$, then $f(a) \in f(T)$. Thus, in any case we have $f(a) \in f(S) \cup f(T)$. Hence, $f(S \cup T) \subseteq f(S) \cup f(T)$.

Conversely, suppose that $b \in f(S) \cup f(T)$. Then by definition, $b \in f(S)$ or $b \in f(T)$ or both. If $b \in f(S)$, then by definition $b = f(a)$ for some $a \in S$. Similarly, if $b \in f(T)$ then by definition $b = f(a)$ for some $a \in T$. So in every case, we have $b = f(a)$ for some $a \in S \cup T$ and by definition $b \in f(S \cup T)$.

Since we've shown both inclusions, we have therefore shown that $f(S \cup T) = f(S) \cup f(T)$, as desired. □

Problem §2.3: 44(b): Let f be a function from A to B. Let S and T be subsets of B. Show that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Solution. Again, we'll show the desired set equality by showing that each set is a subset of the other.

First, consider $a \in f^{-1}(S \cap T)$. By definition, $f(a) \in S \cap T$ and therefore either $f(a) \in S$ and $f(a) \in T$. The fact that $f(a) \in S$ means $a \in f^{-1}(S)$. Similarly, the fact that $f(a) \in T$ means that $a \in f^{-1}(T)$. Hence, by definition $a \in f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Conversely, consider $a \in f^{-1}(S) \cap f^{-1}(T)$. By definition, $a \in f^{-1}(S)$ and $a \in f^{-1}(S)$. From the definition of the preimage of a set, we know that $a \in f^{-1}(S)$ means that $f(a) \in S$. Similarly, $a \in f^{-1}(T)$ means that $f(a) \in T$. As such, we have $f(a) \in S \cap T$ and therefore $a \in f^{-1}(S \cap T)$. Thus, $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

Because we've shown both inclusions, we have therefore shown that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, as desired. □

Problem §3.1: 2: Determine which characteristics of an algorithm described in the text the following procedures have and which they lack.

(a)
```
procedure double(n: positive integer)
while n > 0
    n := 2n
```

(b)
```
procedure divide(n: positive integer)
while n >= 0
    m : = 1/n
    n := n-1
```

(c)
```
procedure sum(n: positive integer)
sum := 0
while i < 10
    sum := sum + i
```

(d)
```
procedure choose(a,b: integers)
x := either a or b
```

Solution. For the sake of concision, we'll just explain the properties that each procedure lacks.

(a) This procedure has every listed characteristic except **finiteness** because the while loop will continue indefinitely. To see this, observe that if the condition for the while loop is met (that $n > 0$), then doubling n will again produce a number greater than zero and the while loop will execute again.

(b) This procedure has every listed characteristic except **effectiveness**. The problem is that the $n := 1/n$ step is not defined when $n = 0$. Because the procedure begins with a positive integer and then uses the while loop to subtract one until n becomes negative, at some point it will reach the $n = 0$ case and encounter this division by zero.

(c) This procedure has every listed characteristic except **definiteness**. The "while $i < 10$" step is not well-defined because the value of i is never set. So a reader trying to execute the procedure would become confused and unable to proceed at that point.

(d) This procedure also has every listed property except **definiteness** because the it doesn't actually tell us whether to set x equal to a or b. So if two people executed this procedure independently, they might end up with different values for x!

□

Problem §3.1: 24: Describe an algorithm that determines whether a function from a finite set to another finite set is one-to-one.

Solution. So we have some notation to work with, let's consider a function $f : A \to B$ where $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$. The input to the algorithm consists of all $n + m$ elements of A and B and the function f. At the beginning of the algorithm, we'll initialize a list of length m called *hit* which tracks which elements of B are images of elements of A. Until we find an element of A with image b_i, the i^{th} entry of *hit* is 0. Once we find such an element, we set the i^{th} entry to 1. The algorithm runs through the elements of A, computing the image of each one and checking to see if the i^{th} entry of *hit* is 0 or 1. If the entry is already 1, then the function is not one-to-one and the algorithm terminates. If the algorithm is not halted prematurely, then we conclude the function is one-to-one. □

Problem §3.1: 52(a,d): Use the greedy algorithm to make change using quarters, dimes, nickels, and pennies for

(a) 87 cents.

(d) 33 cents.

Solution. Recall that the greedy algorithm makes change by selecting the largest coin whose value does not exceed the amount of change to be given, adding that coin to the pile of change, and then decreasing the amount of change required.

(a) The algorithm first uses the maximum number of quarters possible, 3. This leaves $87-3(25) = 87-75 = 12$ cents remaining. It then uses the maximum possible number of dimes, 1, leaving $12-1(10) = 2$ cents. The algorithm cannot use any nickels, because $5 > 2$. Finally, it uses the maximum possible number of pennies, 2, which brings the amount of change required to 0. The algorithm then terminates.

(d) The algorithm first uses the maximum number of quarters possible, 1, leaving $33-25 = 8$ cents. The algorithm cannot use any dimes, because $10 > 8$. Next, it uses the maximum number of nickels possible, 1, leaving $8-1(5) = 3$ cents. Finally, it uses the maximum number of pennies possible, 3, bringing the amount of change remaining to 0. At this point, the algorithm terminates.

$\square$

Problem §3.1: 54(a,d): Use the greedy algorithm to make change using quarters, dimes, and pennies (but no nickels) for

(a) 87 cents.

(d) 33 cents.

Solution. Now, we run the greedy change algorithm again without nickels.

(a) Again, the algorithm begins by using the maximum possible number of quarters, 3, and leaves $87-3(25) = 12$ cents. It then uses the maximum possible number of dimes, 1, which leaves $12-10 = 2$ cents. Unlike in 52, there are no nickels available so it does not test whether or not it's possible to use a nickel. It then uses the maximum number of pennies, 2, as that is the only remaining coin. This brings the amount of change required to 0 and the algorithm terminates.

Note that we reached the same answer as in 52(a).

(d) Again, the algorithm begins by using the maximum possible number of quarters, 1, which leaves 8 cents. It cannot use any dimes, because $10 > 8$. Because there are no nickels available, it then uses the maximum possible number of pennies, 8, leaving 0 cents. The algorithm then terminates.

Observe that in this case the greedy algorithm required a total of nine coins (one quarter, eight pennies). We could have instead used just six coins by using three dimes and three pennies. As such, this example shows that the greedy change algorithm does not produce an optimal solution for this set of coins.

$\square$