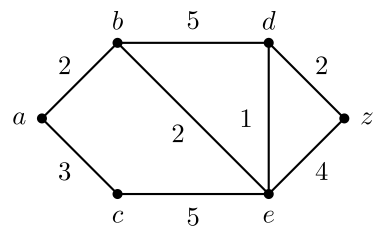
Problem §10.6 - 3: Find the length of a shortest path between a and z in the given weighted graph.



Solution. We use Dijkstra's algorithm to write down the distance from a to each vertex in the graph.

- The closest vertex to a is b, and the distance from a to b is 2. This is just the edging coming off a with smallest weight.
- The next closest vertex is c, with distance 3. This is obtained by extending our known distances by the new edges ac (3), bd (2+5=7), and be (2+2=4). For example, adding bd gets a length of 7 because we already know we need a length of 2 to get from a to b, and we add the weight of 5 from the edge bd. Now out of these options, the smallest value is 3, obtained by adding ac. So we conclude that we have found the distance to c.
- The next closest vertex is e, and the distance is 4. This is obtained by extending our known distances by the new edges bd (2+5=7), be (2+2=4), and ce (3+5=8). The smallest value is 4 at be.
- The next closest vertex is d, and the distance is 5. This is obtained by extending our known distances by the new edges bd (2+5=7), ed (4+1=5), and ez (4+4=8). The smallest value is 5 at ed.
- The next closest vertex is z, and the distance is 7. This is obtained by extending our known distances by the new edges dz (5+2=7) and ez (4+4=8). The smallest value is 7 at dz.

Since we have reached our desired vertex z, we can stop and conclude that the length of the shortest path is 7. □

Problem §10.6 - 5: Find a shortest path between a and z in the weighted graph in Exercise 3.

Solution. The shortest path is constructed by working backward through the way z was obtained in the previous exercise. Vertex z was reached in the last step, by the edge dz. Vertex d was reached one step earlier, by the edge ed. Vertex e was reached one step earlier, by the edge be. Vertex b was reached in the first step, by the edge ab. Now follow these edges. As a result, the shortest path is $abedz$, which has length $2 + 2 + 1 + 2 = 7$. □

Problem §10.6 - 6: Find the length of a shortest path between these pairs of vertices in the weighted graph in Exercise 3.

(a) a and d

(d) b and z

Solution. (a) was already done in problem 3. We found the distance from a to d to be 5. For (d), we follow the same procedure as problem 4. The results are:

- The closest vertex to b is a tie between a and e, with distance 2.
- The next closest is d, with distance 3 and shortest path bed.

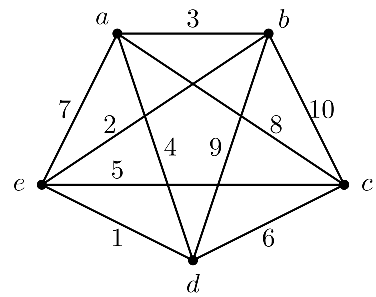
- The next closest is a tie between c and z at distance 5, with shortest paths bac and $bedz$.

So the distance from b to z is 5. ☐

Problem §10.6 - 7: Find shortest paths in the weighted graph in Exercise 3 between the pairs of vertices in Exercise 6 (parts a and d).

Solution. This is computed while computing the distance, as in problems 4 and 5. The shortest path from a to d is $abcd$, and the shortest path from b to z is $bedz$. ☐

Problem §10.6 - 26: Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.

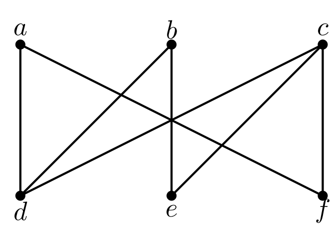


Solution. We make a table of the 12 Hamilton circuits with their edge weights and total weights:

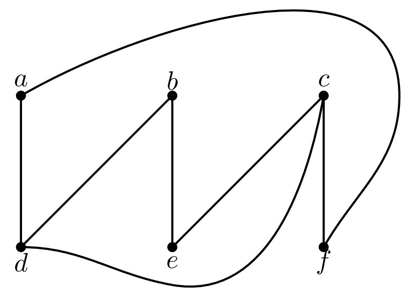
Circuit	Edge Weights	Sum
$abcde$	$3+10+6+1+7$	27
$abced$	$3+10+5+1+4$	23
$abdce$	$3+9+6+5+7$	30
$abdec$	$3+9+1+5+8$	26
$abecd$	$3+2+5+6+4$	20
$abedc$	$3+2+1+6+8$	20
$acbde$	$8+10+9+1+7$	35
$acbed$	$8+10+2+1+4$	25
$acdbe$	$8+6+9+2+7$	32
$acebd$	$8+5+2+9+4$	28
$adbce$	$4+9+10+5+7$	35
$adcbe$	$4+6+10+2+7$	29

The smallest total weight is 20, which is achieved by the circuit $abecd$. ☐

Problem §10.7 - 6: Determine whether the given graph is planar. If so, draw it so that no edges cross.



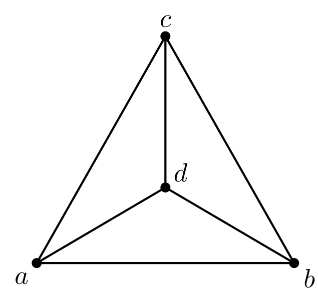
Solution. This graph is indeed planar. Here is the graph drawn so that no edges cross:



□

Problem §10.7 - 11: Show that K_5 is nonplanar using an argument similar to that given in Example 3.

Solution. Label the vertices of K_5 as a, b, c, d, e, and suppose we draw all the edges without crossings. Then the circuit $abca$ makes a simple closed curve in the plane, which divides the plane into two regions (e.g. the inside and outside of a triangle). Now vertex d is either in the inside or outside region. Without loss of generality, suppose d is on the inside (a symmetric argument works if d is on the outside). Then the edges da, db, and dc split the inside region into three subregions. We now have 4 total triangular regions, including the outside:



Now consider vertex e. We must place it inside one of these four triangular regions. For example, suppose we put e inside the triangle acd. Then since e is inside this triangle, and b is outside it, we cannot draw the edge from e to b without making a crossing.

A similar argument works no matter what region we put e in. In any case, the region containing e is a triangle, so it includes 3 of the vertices $\{a, b, c, d\}$. But then the remaining vertex among $\{a, b, c, d\}$ cannot connect to e without crossing the border of the triangle.

□

Problem §10.7 - 13: Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

Solution. Let v, e, and r be the numbers of vertices, edges, and regions, respectively. By Euler's Formula, $r = e - v + 2$. We are given $v = 6$. We can also find e using the handshaking theorem. That is, since all of the degrees are 4, the sum of the degrees is $6 \times 4 = 24$. This is also twice the number of edges, implying that $e = 12$.

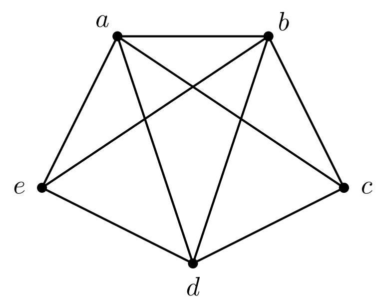
So $v = 6$ and $e = 12$, and plugging these in gets $r = 12 - 6 + 2 = 8$. □

Problem §10.7 - 14: Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

Solution. Let v, e, and r be the numbers of vertices, edges, and regions, respectively. By Euler's Formula, $r = e - v + 2$. We are given $e = 30$ and $r = 20$. Plugging these in gets $20 = 30 - v + 2$, and solving this gets $v = 12$. □

Problem §10.8 - 1: Construct the dual graph for the map shown. (See Rosen for the map!) Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

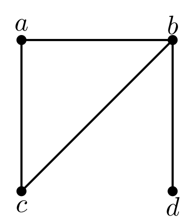
Solution. Here is the dual graph:



The chromatic number of this graph (and, equivalently, of the original map) is 4. We need at least 4 colors since a, b, c, and d are all adjacent, and therefore require 4 different colors. This means that the chromatic number is at least 4.

On the other hand, if we color vertex a red, vertex b blue, vertex d green, and vertices c and e orange, that is a proper 4-coloring of the graph, so the chromatic number is at most 4. Combining the two inequalities, we see that the chromatic number is exactly 4. □

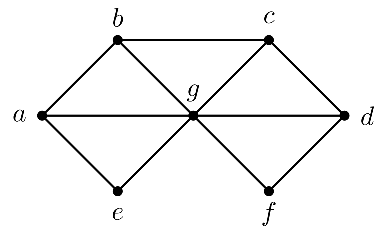
Problem §10.8 - 5: Find the chromatic number of the given graph.



Solution. The chromatic number is 3. We need at least 3 colors because of the triange abc: these three vertices are all adjacent, and so they need 3 different colors.

On the other hand, we can obtain a valid 3-coloring by (for example) coloring a red, b blue, c green, and d red (or green). So, since 3 colors are both necesssary and attainable, the chromatic number is 3. □

Problem §10.8 - 6: Find the chromatic number of the given graph.



Solution. The chromatic number is 3. We need at least 3 colors because the graph contains triangles. For example, a, b, and g are all adjacent to each other, and so they need 3 different colors.

On the other hand, there is a valid 3-coloring, obtained by alternating between two colors around the outside and coloring g a third color. For example, color a red, b blue, c red, d blue, f red, and g green. □

Problem §10.8 - 15: What is the chromatic number of W_n?

Solution. The chromatic number is 3 if n is even, and 4 if n is odd. (Recall that W_n is only defined for $n \geq 3$, so we assume $n \geq 3$.).

Recall that W_n consists of a simple circuit of length n, which we will call the 'outer circuit' of 'outer vertices'; and an additional 'middle' vertex adjacent to all of the outer ones.

Now notice that W_n contains a triangle (e.g. two consecutive outer vertices and the middle one), so at least three colors are needed. If n is even, then in fact three colors suffice: color the outer cicuit by alternating red/blue/red/blue etc., and finally color the middiel vertex green. This shows that the chromatic number is exactly 3 when n is even.

Now assume n is odd. The same coloring above does not work, because the first and last vertices of the outer circuit will get the same color. In fact, this can be turned into a proof that W_n can only be colored with 4 or more colors. First we observe that we cannot 2-color the outer circuit, because if so, the colors would have to alternate, and since n is odd, we end up with two adjacent vertices of the same color (see also the next exercise). So we need 3 colors on the outer circuit alone. And then the middle vertex has to get a different color than each outer vertex (because it is adjacent to all outer vertices), so we are forced to use a fourth color.

On the other hand, 4 colors is possible: start by alternating red/blue/red/blue around the outside circuit, and coloring the middle vertex green. This results in two adjacent red vertices on the outside: now just change either of these to a fourth color, say purple, and we obtain a valid 4-coloring. This shows that the chromatic number is 4 when n is odd. □

Problem §10.8 - 16: Show that a simple graph that has a circuit with an odd number of vertices in it cannot be colored using two colors.

Solution. This is outlined in the solution to the previous exercise. Suppose the vertices in the circuit are $v_1, ..., v_{2n+1}, v_1$ (where we use $2n+1$ because the length is odd), and suppose we manage to color the graph with two colors, say red and blue. Without loss of generality, we can assume v_i is colored red (otherwise, we can make a symmetric argument with red and blue reversed). Then following around the circuit, v_2 has to be colored blue, v_3 has to be colored red, and so on. In general, each v_k is colored red if k is odd and blue if k is even.

Now the last vertex in the circuit, v_{2n+1}, is colored red (since $2n+1$ is odd). But then v_{2n+1} and v_1 are adjacent red vertices, which invalidates the coloring. Thus our coloring fails, and we reach a contradiction. □