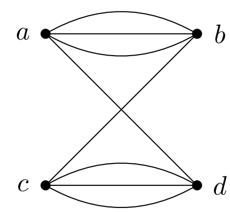
Problem §10.3 - 14: Represent the following graph using an adjacency matrix.

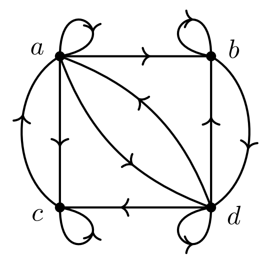


Solution. If we order the vertices alphabetically, this graph corresponds to the adjacency matrix

$$A = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

□

Problem §10.3 - 20: Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

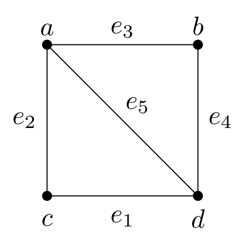


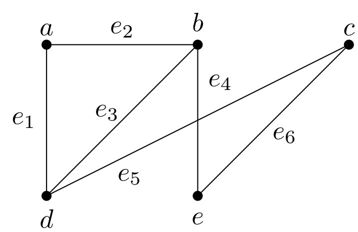
Solution. Ordering the vertices in alphabetic order, the corresponding adjacency matrix is:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

□

Problem §10.3 - 26: Use incidence matrices to represent the following graphs:





Solution. Let's begin with the graph shown on the left. Ordering the vertices alphabetically as a, b, c, d and the edges in numerical order as e_1, e_2, e_3, e_4, e_5, we obtain the incidence matrix

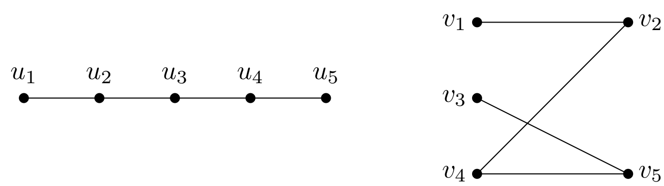
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Next, let's consider the graph shown on the right. Again, let's order the vertices alphabetically as a, b, c, d, e and the edges numerically as $e_1, e_2, e_3, e_4, e_5, e_6$. We obtain the incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

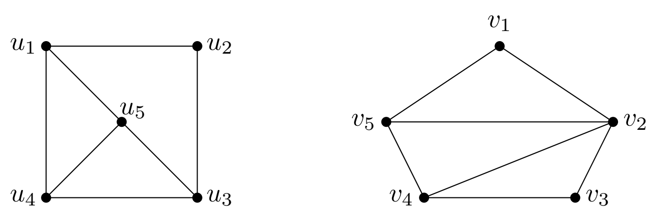
□

Problem §10.3 - 34: Determine if the given pair of graphs is isomorphic. Either exhibit an isomorphism or provide a rigorous argument that none exists.



Solution. These graphs **are isomorphic** - both are ways to draw P_5. One possible graph isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$. □

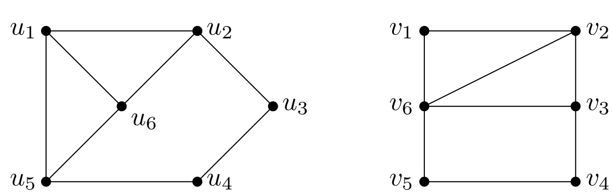
Problem §10.3 - 36: Determine if the given pair of graphs is isomorphic. Either exhibit an isomorphism or provide a rigorous argument that none exists.



Solution. These graphs are **not isomorphic**. To see that these graphs cannot be isomorphic, observe that the graph on the right contains a vertex, v_2, of degree 4, whereas the maximum vertex degree in the graph on the left is 3.

(Recall that in order to show that two graphs are not isomorphic, you simply need to find at least one graph invariant that is different.) □

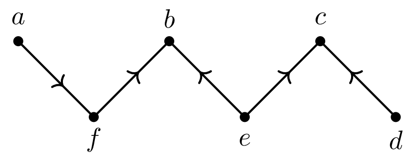
Problem §10.3 - 40: Determine if the given pair of graphs is isomorphic. Either exhibit an isomorphism or provide a rigorous argument that none exists.



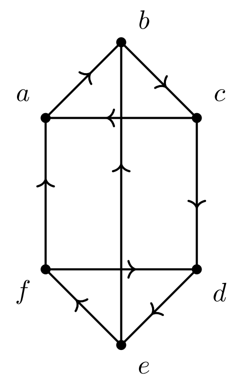
Solution. These graphs are **not isomorphic**. Once again, the graph on the right contains a vertex, v_6, of degree 4, whereas the maximum vertex degree in the graph on the left is 3. □

Problem §10.4 - 12: Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

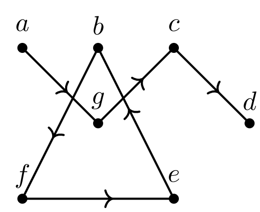
(a)



(b)



(c)



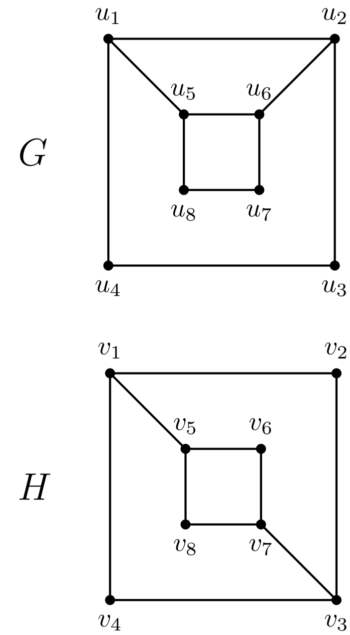
Solution. (a) This graph is not strongly connected because (for instance) there is no path from f to a. However, it is weakly connected because in the underlying graph a, f, b, e, c, d is a path containing all vertices.

(b) This graph is strongly connected because a, b, c, d, e, f, a is an Eulerian circuit, and therefore there is a path from any vertex to any other vertex by following this circuit (e.g. from b to e, we traverse b, c, d, e, and from e to b, we traverse e, f, a, b).

(c) This graph is not strongly or weakly connected (for instance) there is no path from a to b, even in the underlying graph.

□

Problem §10.4 - 20: Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

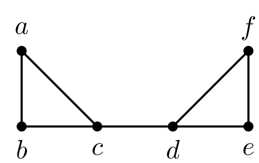


Solution. G and H both have 8 vertices, 10 edges, and degree sequence $3, 3, 3, 3, 2, 2, 2, 2$, so these three invariants agree. However, G has a simple circuit of length 8 $(u_1, u_5, u_8, u_7, u_6, u_2, u_3, u_4, u_1)$, but H does not not have such a circuit, as we will show.

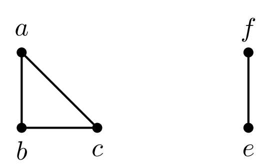
If H did have a simple circuit of length 8, every vertex would appear exactly once (or exactly twice if it's the first and last vertex). Since v_2 has degree 2, it must appear directly after v_1 and directly before v_3, or vice-versa. Similarly, since v_4 has degree 2, it must appear directly after v_1 and directly before v_3, or vice-versa. Therefore, v_1 must appear directly after v_2 and directly before v_4, or vice-versa, and similarly for v_3. Thus, starting the circuit from v_1 we must have v_1, v_2, v_3, v_4 or the reverse, but then we're back to v_1 and so the other four vertices cannot appear.

(There is simpler way to show that G and H aren't isomorphic, which doesn't involve paths. G has an edge between u_3 and u_4, which are both vertices of degree 2. If f were an isomorphism from G to H, both $f(u_3)$ and $f(u_4)$ would need to be degree-2 vertices, and they would need to be adjacent since u_3 and u_4 are adjacent, but H doesn't have any edges whose endpoints are both vertices of degree 2, so this is impossible.) □

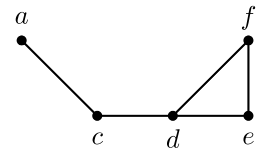
Problem §10.4 - 32: Find all the cut vertices of the given graph.



Solution. The cut vertices are c and d. Removing either of them will remove the edge in the middle, and there will no longer be a path from b to e. For instance, if d is removed, the resulting graph will be the following graph, which is disconnected:

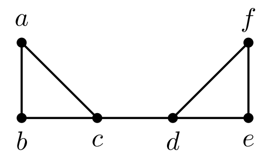


On the other hand, removing any other vertex will not disconnect the graph. For instance, if b is removed, the resulting graph will be the following graph, which is still connected:

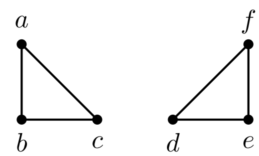


□

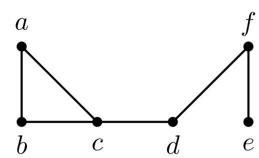
Problem §10.4 - 34: Find all the cut edges of the given graph.



Solution. The only cut edge is the edge from c to d. If that edge is removed, the remaning graph is as follows:

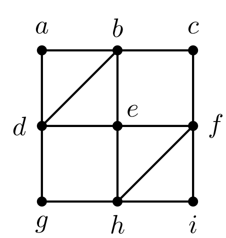


On the other hand, every other edge is part of a simple circuit, so removing it leaves the graph connected. For example, if we remove the edge from d to e, the following connected graph remains:



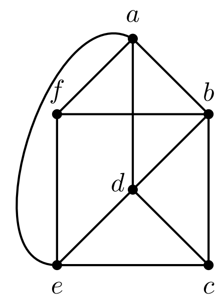
□

Problem §10.5 - 2: Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution. The degree sequence of this graph is $4, 4, 4, 4, 4, 2, 2, 2, 2$, and since all degrees are even, the graph must have an Eulerian circuit. One example is $a, d, b, e, d, g, h, i, f, h, e, f, c, b, a$. □

Problem §10.5 - 4: Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution. The degree sequence of this graph is $4, 4, 4, 4, 3, 3$, and since exactly two degrees are odd, the graph must have an Eulerian path but not an Eulerian circuit. One example of an Eulerian path is $c, b, d, c, e, d, a, b, f, e, a, f$. □

Problem §10.5 - 26: For which values of n do these graphs have an Euler circuit?

(a) K_n

(b) C_n

(c) W_n

(d) Q_n

Solution. (a) Every vertex in K_n has degree $n-1$, which is even if and only if n is odd. Therefore, K_n has an Eulerian circuit if and only if n is odd.

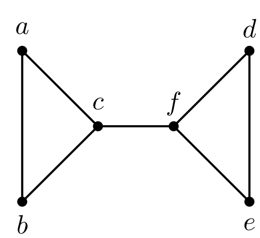
(b) Every vertex in C_n has degree 2. Therefore, C_n has an Eulerian circuit for all n.

(c) Every vertex of W_n other than the center one has degree 3. Since the family W_n starts at $n = 3$, there are always at least 3 vertices of odd degree, so there is never an Eulerian circuit for any n

(d) Every vertex in Q_n has degree n. Therefore, Q_n has an Eulerian circuit if and only if n is even.

□

Problem §10.5 - 30: Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists



Solution. Because the degrees of the vertices a and b are both two, every edge incident with these vertices must be part of any Hamilton circuit. On the other hand, the edge between c and f must be part of any Hamiltonian circuit since it is a cut edge and so every walk containing both c and

f must contain this edge. It is now easy to see that no Hamilton circuit can exist in the graph, for any Hamilton circuit would have to contain three edges incident with c, which is impossible.

(More simply, no graph with a cut edge can contain either a Hamiltonian circuit or an Eulerian circuit since removing it leaves a disconnected graph with no Eulerian/Hamiltonian path.) $\square$