Problem §2.1: 10(a,c,e,g): Determine whether the following statements are true or false.

(a) $\emptyset \in \{\emptyset\}$

(c) $\{\emptyset\} \in \{\emptyset\}$

(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Solution. (a) True.

(c) False. The set $\{\emptyset\}$ has only one element, $\emptyset$, and is not a subset of itself.

(e) True.

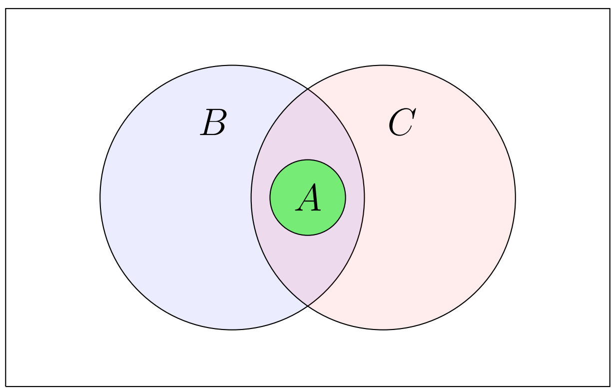
(g) False, $\{\{\emptyset\}\}$ and $\{\{\emptyset\}, \{\emptyset\}\}$ are both representations of the same set and therefore $\{\{\emptyset\}\}$ is not a **proper** subset of $\{\{\emptyset\}, \{\emptyset\}\}$. It would be true, however, if we wrote $\{\{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset\}\}$.

N.B.: This is in the same spirit as the difference between a strict inequality, $<$*, and a weak inequality,* $\leq$*. In that more familiar context, we know that* $1 \leq 1$*, but* $1 \nless 1$*.*

□

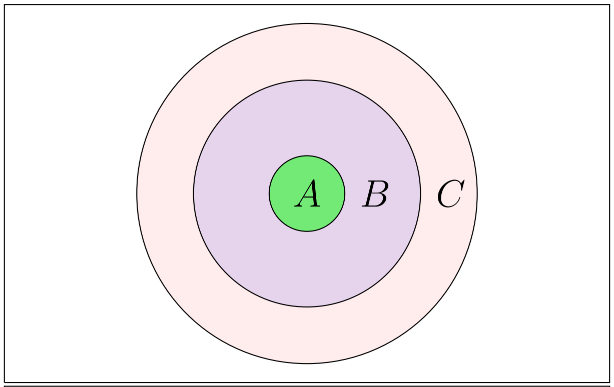
Problem §2.1: 16: Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

Solution. Most generically, we can represent these relationships with the following Venn diagram:

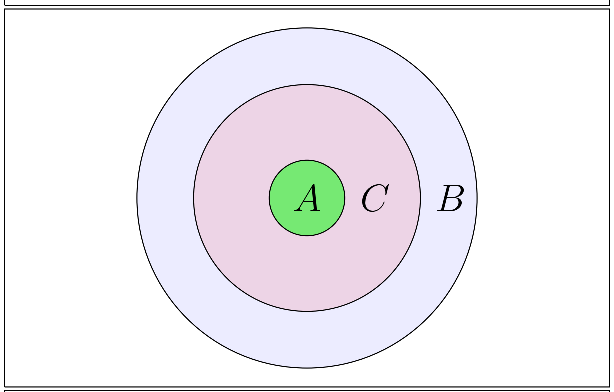


To understand why this is the correct representation, note that the problem statement doesn't directly tell us anything about the relationship between sets B and C. Because $A \subset B$ and $A \subset C$, we know that sets B and C must share all of the elements of A. We don't know, however, if B and C share other elements or if in fact $A = B \cap C$. Because A is a proper subset of both B and C, we know that $A \neq B$ and $A \neq C$. Hence, both B and C must have at least one element that is not in A - i.e., $B \backslash A \neq \emptyset$ and $C \backslash A \neq \emptyset$. We don't know, however, if this is a shared element.

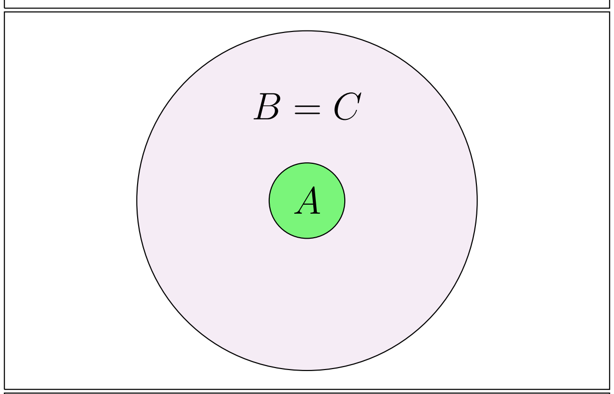
In fact, we could have any of the following special cases:



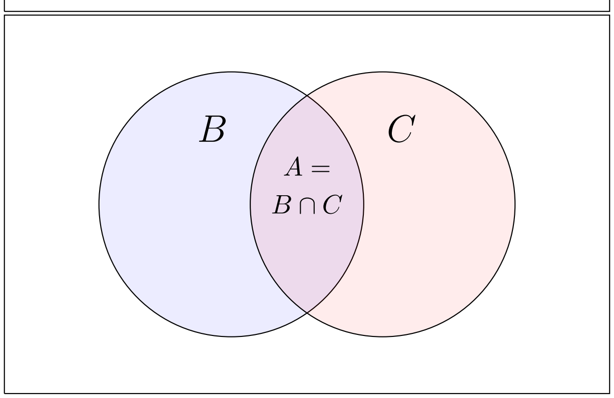
$A \subset B \subset C$



$A \subset C \subset B$



$A \subset B = C$



$A = B \cap C$

$\square$

Problem §2.1: 20: What is the cardinality of each of the following sets?

(a) $\emptyset$

(b) $\{\emptyset\}$

(c) $\{\emptyset, \{\emptyset\}\}$

(d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Solution. (a) By definition, $|\emptyset| = 0$.

(b) $|\{\emptyset\}| = 1$

(c) $|\{\emptyset, \{\emptyset\}\}| = 2$

(d) $|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| = 3$

□

Problem §2.1: 26: Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Solution. To show that $A \times B \subseteq C \times D$, we need to show that every element of $A \times B$ is also an element of $C \times D$. From the definition of a Cartesian product, we know that elements of $A \times B$ are ordered pairs (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$. Likewise, the fact that $B \subseteq D$ means that $b \in D$. Hence, (a, b) is an ordered pair with $a \in C$ and $b \in D$ and is therefore by definition an element of the Cartesian product $C \times D$. □

Problem §2.1: 32(a,c): Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find the following Cartesian products.

(a) $A \times B \times C$

(c) $C \times A \times B$

Solution. Using the definition of the Cartesian product, we compute:

$$A \times B \times C = \left\{ \begin{array}{c} (a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1) \\ (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1) \\ (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1) \end{array} \right\}$$

$$C \times A \times B = \left\{ \begin{array}{c} (0, a, x), (0, a, y), (0, b, x), (0, b, y) \\ (0, c, x), (0, c, y), (1, a, x), (1, a, y) \\ (1, b, x), (1, b, y), (1, c, x), (1, c, y) \end{array} \right\}$$

□

Problem §2.2: 4: Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find:

(a) $A \cup B$.

(b) $A \cap B$.

(c) $A - B$.

(d) $B - A$.

Solution. (a) $A \cup B = \{a, b, c, d, e, f, g, h\} = B$. This is a specific instance of the more general fact that if $A \subseteq B$, then $A \cup B = B$.

(b) $A \cap B = \{a, b, c, d, e\}$. Again, this is a specific instance of the more general fact that if $A \subseteq B$, then $A \cap B = A$.

(c) $A - B = \emptyset$. This is a specific instance of the more general fact that if $A \subseteq B$, then $A - B = \emptyset$.

(d) $B - A = \{f, g, h\}$.

$\square$

Problem §2.2: 14: Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

Solution. To find A, we'll use the fact that $A = (A \cap B) \cup (A - B)$. In other words - we can divide the set of elements in A into two sets: elements that are also in B (i.e., $A \cap B$) and elements that aren't in B (i.e., A-B). Similarly, to find B we use the fact that $B = (B \cap A) \cup (B - A)$. Doing so, we find that

$$\begin{aligned} A &= (A \cap B) \cup (A - B) \\ &= \{3, 6, 9\} \cup \{1, 5, 7, 8\} \\ &= \{1, 3, 5, 6, 7, 8, 9\} \\ B &= (B \cap A) \cup (B - A) \\ &= \{3, 6, 9\} \cup \{2, 10\} \\ &= \{2, 3, 6, 9, 10\} \end{aligned}$$

$\square$

Problem §2.2: 15: Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (a) showing each side is a subset of the other side and (b) by using a membership table.

Solution. (a) To show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$, we'll show both that each set is a subset of the other.

We'll begin by showing that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Suppose $x \in \overline{A \cup B}$. Then by definition of the complement of a set, $x \notin A \cup B$ and therefore $x \notin A$ and $x \notin B$. Equivalently, we could write $x \in \overline{A}$ and $x \in \overline{B}$. Hence, $x \in \overline{A} \cap \overline{B}$ by definition and therefore $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Next, we'll show the other inclusion, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Suppose $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. By definition, this is equivalent to writing that $x \notin A$ and $x \notin B$. Because the union $A \cup B$ consists of elements in at least one of the sets A and B, this means that $x \notin A \cup B$ or equivalently $x \in \overline{A \cup B}$. Hence, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Because we've shown both inclusions, we've shown that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ as desired.

(b) To prove this identity, we can create the following membership table:

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Because the columns for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ are identical, we conclude that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$\square$

Problem §2.2: 24: Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

Solution. We know two methods for proving set identities - (1) proving that each set is a subset of the other and (2) membership tables. Although you don't need to show both methods (proving that something is true once is sufficient!), I'll show both here for the sake of completeness.

Method (1):

First, we'll show that $(A-B)-C \subseteq (A-C)-(B-C)$. Consider $x \in (A-B)-C$. By definition, this means that $x \notin C$ and $x \in A - B$. Similarly, $x \in A - B$ means that $x \notin B$ and $x \in A$. So altogether we've deduced that $x \in A$ and $x \notin B, C$. Hence, by definition $x \in A - C$ and $x \notin B - C$. Therefore, $x \in (A - C) - (B - C)$ and we have $(A - B) - C \subseteq (A - C) - (B - C)$.

Next, we'll show that $(A - C) - (B - C) \subseteq (A - B) - C$. Suppose $x \in (A - C) - (B - C)$. Then by definition $x \in A - C$ and $x \notin B - C$. By definition, $x \in A - C$ means that $x \in A$ and $x \notin C$. Together, $x \notin B - C$ and $x \notin C$ imply that $x \notin B$ (this is because if we had $x \in B$ and $x \notin C$, then we would have $x \in B - C$ rather than $x \notin B - C$). So altogether, we know that $x \in A$ and $x \notin B, C$. Thus, by definition $x \in A - B$ and therefore $x \in (A - B) - C$. This gives us the desired inclusion, $(A - C) - (B - C) \subseteq (A - B) - C$.

Having shown both inclusions, we conclude that $(A - C) - (B - C) = (A - B) - C$ as desired.

Method (2):

A	B	C	$A - B$	$A - C$	$B - C$	$(A - B) - C$	$(A - C) - (B - C)$
1	1	1	0	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	1	0	0	0	0
0	1	1	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	0	0	1	0	0
1	0	0	1	1	0	1	1
0	0	0	0	0	0	0	0

Because the columns for $(A - B) - C$ and $(A - C) - (B - C)$ are identical, we conclude that $(A - B) - C = (A - C) - (B - C)$, as desired. $\square$

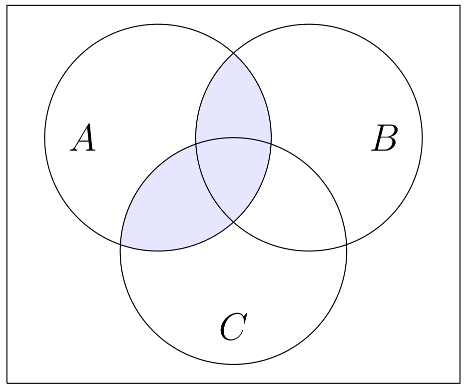
Problem §2.2: 26: Draw the Venn diagrams for each of the following combinations of the sets A, B, and C.

(a) $A \cap (B \cup C)$

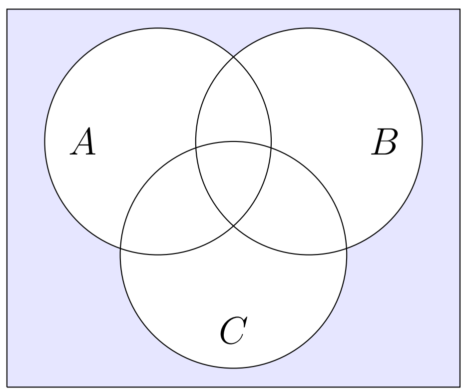
(b) $\overline{A} \cap \overline{B} \cap \overline{C}$

(c) $(A - B) \cup (A - C) \cup (B - C)$

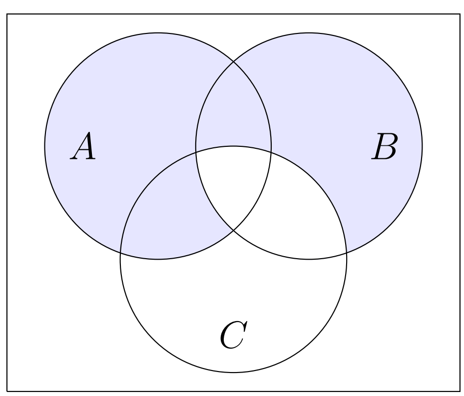
Solution. (a) Generically, we can draw a Venn diagram for $A \cap (B \cup C)$ as:



(b) Generically, we can draw a Venn diagram for $\overline{A} \cap \overline{B} \cap \overline{C}$ as:



(c) Generically, we can draw a Venn Diagram for $(A - B) \cup (A - C) \cup (B - C)$ as:



$\square$