Solutions to Math 213-X1 Final Exam — Dec. 19, 2024

1. (12 points) Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$, and $C = \{4, 5\}$. Write the following sets, using proper mathematical notation. *(No work necessary for this problem!)*

   (a) (3 points) $A \cup (B \cap C)$

   $\{1, 2, 3, 4\}$

   (b) (3 points) $\mathcal{P}(B)$

   $\{\emptyset, \{3\}, \{4\}, \{3, 4\}\}$

   (c) (3 points) $B \times A$

   {(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)}

   (d) (3 points) $A - (B - C)$ (also written as $A \setminus (B \setminus C)$).

   $B \setminus C = \{3\}$, so $A \setminus (B \setminus C) = \{1, 2\}$

2. (15 points) Let f be a function from A to B, and let $S \subseteq A$.

   (a) (10 points) Prove that for all $S \subseteq A$ that $S \subseteq f^{-1}(f(S))$.

   Let $x \in S$. We show that $x \in f^{-1}(f(S))$. Let $y = f(x)$. Then $y \in f(S)$ since $x \in S$. By definition,
   $$f^{-1}(f(S)) = \{z \in A | f(z) \in f(S)\}.$$

   (b) (5 points) Give an example of A, B, S, and f such that $f^{-1}(f(S)) \not\subseteq S$.

   Let $A = \{1, 2\}, B = \{3\}, S = \{1\}$, and let f be the function $f : A \to B$ defined by $f(1) = f(2) = 3$. Then $f(S) = f(\{1\}) = \{3\}$, and $f^{-1}(f(S)) = f^{-1}(\{3\}) = \{1, 2\}$, which is not a subset of S.

3. (10 points) Prove that for all positive integers n,
$$1^2 + 2^2 + 3^3 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Note: 3^3 should read 3^2. This was corrected during the exam.)

Let $P(n)$ be the statement
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We prove that $P(n)$ is true for all n by induction on n. Base case: If $n = 1$, then
$$1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6},$$
so $P(1)$ is true.

Induction step: Assume that $P(k)$ is true for some $k \geq 1$. We prove that $P(k+1)$ is true.

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},
\end{aligned}$$

where the first equality uses the inductive hypothesis. Thus, $P(k+1)$ is true, and therefore $P(n)$ is true for all n by induction.

4. (15 points) Let a_n be the number of ways to cover a $1 \times n$ board with any number of:

- Red tiles (R) of length 1
- Blue tiles (B) of length 2

such that no two blue tiles are adjacent to each other. For example, if $n = 5$, then RRRRR is a valid tiling, and so is BRB, but BBR is not since it has two adjacent blue tiles.

(a) (5 points) Find a recurrence relation for a_n

Consider the last tile. Either it is red, and the remaining length-$(n-1)$ board can be covered in any way, or it is blue. In this latter case, the tile adjacent to it must be red because no two blue tiles can be adjacent to each other. The remaining length-$(n-3)$ board can be covered in any way.

Thus, the recurrence relation is

$$a_n = a_{n-1} + a_{n-3}, \qquad n \geq 3.$$

(b) (5 points) What are the initial conditions?

We need enough initial conditions so that the recurrence relations can be applied, that is, we need a_0, a_1, a_2 (a_1, a_2, a_3 would also be fine; either way, you need three initial conditions).

An empty board can be filled in 1 way (left empty). A board of length 1 can be filled in 1 way (R). A board of length 2 can be filled in 2 ways (RR or B). We have

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 2.$$

(c) (5 points) Compute a_8 using your recurrence relation and initial conditions.

Applying the recurrence relation, and starting with the initial conditions,

$$\begin{aligned}
a_3 &= a_2 + a_0 = 2 + 1 = 3 \\
a_4 &= a_3 + a_1 = 3 + 1 = 4 \\
a_5 &= a_4 + a_2 = 4 + 2 = 6 \\
a_6 &= a_5 + a_3 = 6 + 3 = 9 \\
a_7 &= a_6 + a_4 = 9 + 4 = 13 \\
a_8 &= a_7 + a_5 = 13 + 6 = 19
\end{aligned}$$

5. (20 points) For each of the following sets of properties, either draw a graph which has these properties, or explain why no such graph can exist

   (a) (4 points) A bipartite graph which has chromatic number at least 5.

   Such a graph cannot exist since all bipartite graphs have chromatic number (one or) two.

   (b) (4 points) A graph where every vertex has odd degree.

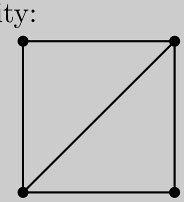
   This is possible (it is only impossible if the number of vertices needs to be *odd*). Here is the smallest graph that works:

   (c) (4 points) A tree with degree sequence $5, 4, 3, 3, 2, 1, 1, 1$.

   This is impossible. By the Handshake Theorem, the sum of the degrees is twice the number of edges. The sum of these degrees is 20, so the graph must have 10 edges. However, there are 8 vertices and trees must have exactly one more vertex than edge.

   (d) (4 points) A graph which has a Hamiltonian circuit and an Eulerian path, but no Eulerian circuit.

   Many graphs work. Here is one possibility:



   (e) (4 points) A digraph which is weakly connected but not strongly connected, and which is the diagraph corresponding to a symmetric relation.

   This is impossible. A digraph for a symmetric relation always has edges come in pairs: if an edge goes from a to b, there is another edge going from b to a. If a digraph is weakly connected, this means that the underlying graph (forgetting the arrows) is connected, so there is a path from every vertex to every other vertex if we are allowed to go against the arrows. But since the edges come in pairs, wherever you go against the arrow, there is another edge between the same pair of vertices where the direction of travel is with the arrow; thus the digraph is strongly connected.

6. (15 points) Prove that for all positive integers n that

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n$$

   (a) (10 points) Using a combinatorial argument.

   This is the general problem of putting n distinguishable objects into 4 distinguishable boxes. Any equivalent set will suffice. We'll use the one below as an example:

   Consider the set of strings of length n with digits chosen from $\{0, 1, 2, 3\}$.

   Method 1: For each digit in turn, choose one of the four possible digits. By the product rule, we get a total of $4 \cdot 4 \cdot \ldots \cdot 4 = 4^n$ total such strings.

   Method 2:

   - Choose the number k of nonzero digits in the string
   - Choose which digits are nonzero – $\binom{n}{k}$ ways to make this choice

- For each of the k nonzero digits, choose one of the three possible nonzero digits. By the product rule, there are $3 \cdot 3 \cdot \ldots \cdot 3 = 3^k$ ways to do this.

Thus, Method 2 gives us a total of $\sum_{k=0}^{n} 3^k \binom{n}{k}$ valid strings.
Since both methods count the same set, the sizes must be equal, so

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n.$$

(b) (5 points) Using the Binomial Theorem.

The Binomial Theorem is the statement that for all $n \geq 0$ and for all x and y,

$$\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k} = (x+y)^n.$$

Plugging in $x = 3, y = 1$ to this formula, we get $x + y = 4$, and the formula becomes

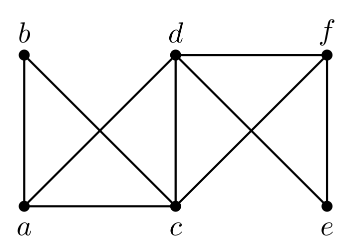
$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n.$$

7. (10 points) Suppose that 2% of people have a certain disease. There is a test for the disease such that 80% of people with the disease test positive and only 10% without the disease test positive. What is the probability that someone who tests positive has the disease?

Let E be the event of having the disease and F be the event of testing positive. We want to compute $P(E|F)$. From the problem statement, we have $P(E) = 0.02, P(\overline{E}) = 0.98, P(F|E) = 0.8, P(F|\overline{E}) = 0.1$. Applying Bayes' Theorem,

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|\overline{E})P(\overline{E})} = \frac{0.8 \cdot 0.02}{0.8 \cdot 0.02 + 0.1 \cdot 0.98} = 0.140 = 14\%.$$

8. (10 points) Determine the chromatic number of the following graph:



The chromatic number is 3. To see this, note that a, b, and c are all adjacent to each other, so must all have different colors, so the chromatic number is at least 3. On the other hand, color a and f red, b and d blue, and c and e green. This is a valid coloring with 3 colors, so the chromatic number is at most 3 (and thus exactly 3).

9. (24 points) For each of the following questions, **only your answer will be graded**.

(For this problem, you may leave your answer in terms of binomial coefficients)

(a) (4 points) What is the smallest positive integer n such that $f(x) = \frac{(x+1)^3(x+2)^2}{x \log x}$ is $O(x^n)$?

The smallest such integer is $n = 4$. The largest term on the top is x^5, and the largest term on the bottom is $x \log x$. Canceling x from top and bottom gives $x^4/\log x$, which is $O(x^4)$ since $\log x > 1$ for large x. On the other hand, $x^4/\log x$ is not $O(x^3)$ since $\log x < x$ for large x.

(b) (4 points) Let $A = \{a, b, c, d\}$ and let R be the following relation on A:

$$R = \{(a, b), (b, c), (c, d), (d, a)\}.$$

Is R reflexive? Symmetric? Antisymmetric? Transitive? (Give your answer to all four of these questions).

R is antisymmetric, but not reflexive, symmetric, or transitive.

(c) (4 points) What is the number of strings of length 8 consisting of decimal digits (0-9) such that no two consecutive digits are equal (for example, 12131482 is a valid string, but 13445032 is not valid since it has consecutive 4's).

There are $10 \cdot 9^7$ such strings. Choose the digits from left to right, and use the product rule There are 10 options for the first digit, and then 9 options for each of the remaining digits since it can't equal the previous digit.

(d) (4 points) How many ways are there to buy 12 cookies from a shop with 5 flavors if you can buy as many as you want of each flavor.

Sticks-and-stones: $\binom{12+5-1}{12} = \binom{16}{12}$.

(e) (4 points) What is the probability that a 5-card poker hand is two pairs (i.e. contains two of one kind, two of a second kind, and one of a third kind).

We choose two of the 13 kinds to have a pair, then one of the remaining 11 kinds to have the remaining card. For each of the pairs, we choose two of the four cards of that kind, and for the fifth card, we choose one of the four cards. Finally, we divide by the total number of hands, to get:

$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}.$$

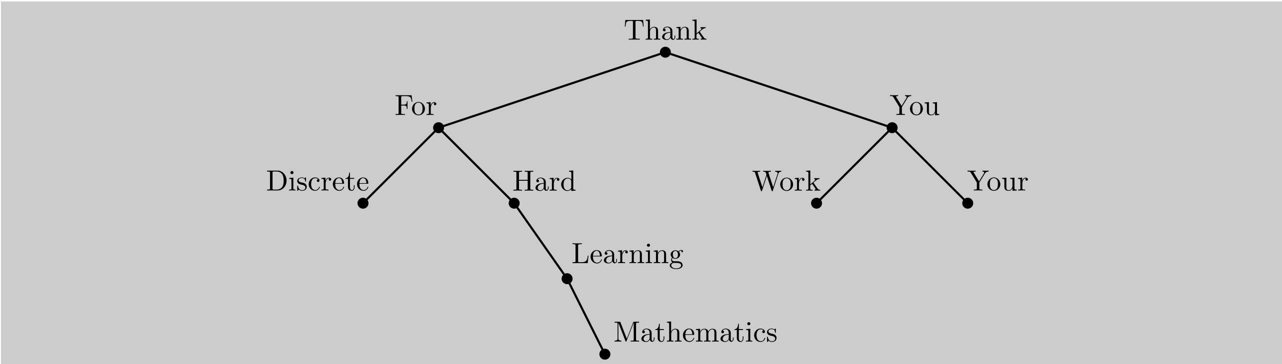
(f) (4 points) If G is a connected planar graph with 20 vertices and 30 edges, how many regions must any planar representation of G have?

It must have 12 regions. We apply Euler's formula: $v - e + r = 2$. Plug in $v = 20, e = 30$, and solve for r.

10. (10 points) Use the algorithm discussed in class to construct a binary search tree from the following list:

Thank, You, For, Your, Hard, Work, Learning, Discrete, Mathematics,

inputting the words in the order given, and using alphabetical order. *(Note: "you" comes before "your" in alphabetical order).*


Thank
For
You
Discrete
Hard
Work
Your
Learning
Mathematics