Midterm: Wed. Feb. 8th in class

Thm 17 (cont.) K/F finite $\Leftrightarrow$ K/F gen'd by finitely many alg. elts. over F.

$\Rightarrow$: If $[K:F]=n$, choose some element $\alpha_1 \in K \setminus F$.

Then $[K:F(\alpha_1)] = \dfrac{[K:F]}{[F(\alpha_1):F]} < [K:F]$. The result follows by induction.

Cor 18: If α and β are alg. over F, then so are $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$, α/β $(\beta \neq 0)$.

Pf: All of these elts. are in $F(\alpha,\beta)$, which is finite/ by Thm. 17, so by Cor. 13 they are alg.

Cor 19: If L/F is any field ext'n, the subfield of L of alg. elts. $/F$ is a subfield.

Ex: (algebraic numbers)

$\mathbb{C}/\mathbb{Q}$. Let $\overline{\mathbb{Q}}$ be the set of all $z \in \mathbb{C}$ which are alg. $/\mathbb{Q}$. Since x^n-2 is irred $\forall n$, $[\overline{\mathbb{Q}}:\mathbb{Q}] \geq n$, so $[\overline{\mathbb{Q}}:\mathbb{Q}] = \infty$.

§13.3: Straightedge & Compass Constructions

Game:

1) start w/ a line segment of length 1
2) use straightedge & compass to construct other lengths/angles
3) end up w/ desired figure

Comes down to constructing a particular length (ie number)

Three problems (Ancient Greeks)

I) Doubling the cube: Construct a cube w/ volume twice the original cube (construct $\sqrt[3]{2}$)

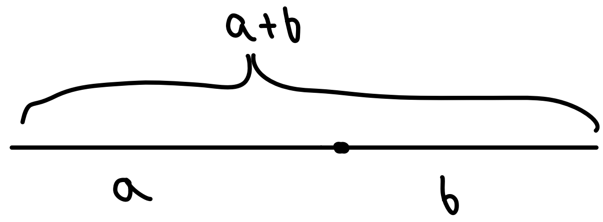
II) Trisecting an angle: Given angle θ, construct angle $\theta/3$ (construct $\cos\frac{\theta}{3}$ given $\cos\theta$)

III) Squaring the circle: Construct a square w/ same area as unit circle (construct π)

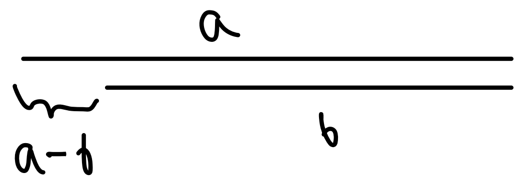
Def: A constructible number is a length which can be constructed via straightedge & compass. Let K be the set of constructible numbers.

If a, b are constructible, can construct:

$a+b$



$a-b$



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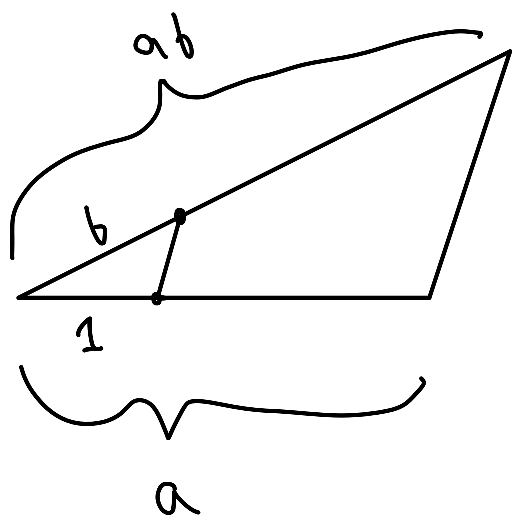

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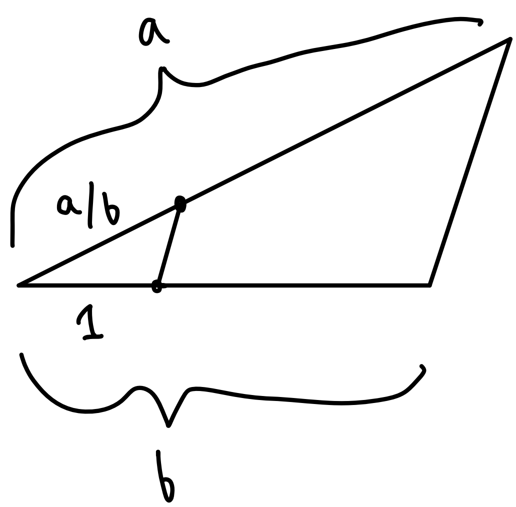
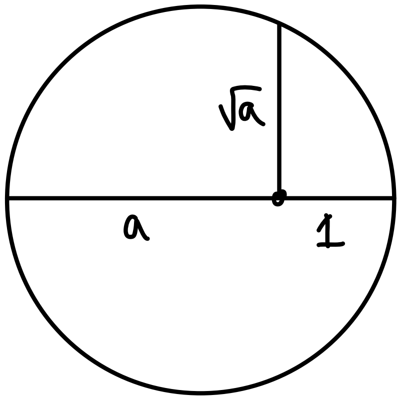


K is a field!

$\mathbb{Q} \subseteq K$

$\sqrt{a}$

(or $\sqrt{b}$)



Thm (who?): Every elt. of K can be obtained from a sequence of the above constructions

Prop 23: If $\alpha \in K$, then $[F(\alpha):F]$ is a power of 2.

Pf: If $\beta \in K$, then $\mathbb{Q}(\beta) \subseteq K$, so let

$$\mathbb{Q} \subseteq \mathbb{Q}(\beta_1) \subseteq \mathbb{Q}(\beta_1, \beta_2) \subseteq \cdots \subseteq \mathbb{Q}(\beta_1, \dots, \beta_n) \ni \alpha,$$

where each β_i is constructed from elts of $\mathbb{Q}(\beta_1, \dots, \beta_{i-1})$ using one of the above constructions. Then either $\beta_i \in$

i.e. $[\mathbb{Q}(\beta_1, \dots, \beta_i) : \mathbb{Q}(\beta_1, \dots, \beta_{i-1})] = 1$ or $\beta \notin \mathbb{Q}(\beta_1, \dots, \beta_{i-1})$, but $\beta^2 \in \mathbb{Q}(\beta_1, \dots, \beta_{i-1})$, in which case $[\mathbb{Q}(\beta_1, \dots, \beta_i) : \mathbb{Q}(\beta_1, \dots, \beta_{i-1})] = 2$.

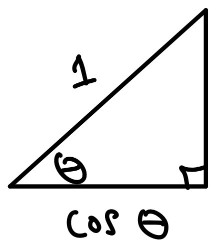
By the Tower law, $[\mathbb{Q}(\beta_1, \dots, \beta_n) : \mathbb{Q}]$, is a power of 2, and again by the Tower law, $[\mathbb{Q}(\alpha):\mathbb{Q}]$ divides, so is a power of 2 itself.

Thm 24: None of I, II, III is possible.

Pf: I) $\sqrt[3]{2}$ is degree 3.

III) π is transcendental i.e. $[\mathbb{Q}(\pi):\mathbb{Q}] = \infty$

II) Trisecting is possible for <u>some</u> angles, just not <u>all</u> angles



Let $\theta = 60°$, so $\cos\theta = \frac{1}{2} \in \mathbb{Q}$.

Triple angle formula:

$$\cos\theta = 4\cos^3 \theta/3 - 3\cos\theta/3$$

So $\cos 20°$ is a root of

$$p(x) = 4x^3 - 3x - \tfrac{1}{2} \leftarrow \text{irreducible}$$

So $\cos 20°$ is deg. 3 over $\mathbb{Q}$ $\Rightarrow$ $\cos 20° \notin K$.

Next time: splitting fields and algebraic closures