H/W 7 posted (due Tues. 3/14)

Final exam: Thurs. 3/23 8:30 - 11:30 Room 200-205 (see email)

Today: Insolvability of the quintic, Cardano's formula
(Abel - Ruffini)

Thm 39: Let $f(x) \in F[x]$ w/ char $F = 0$. Then $f(x)$ can be solved by radicals $\Leftrightarrow$ $Gal(f)$ is solvable.

Pf: Let K be the splitting field of f.

$\Rightarrow$: By Lemma 38, every root of f is contained in a radical Galois ext'n s.t. every radical extn is cyclic, so the composite L is also such a field. Let

$$F \subseteq K_0 \subseteq \cdots \subseteq K_s = L, \quad K_{i+1} = K_i(\sqrt[n_i]{a_i})$$

$\Updownarrow$ Gal. corresp. $\qquad K_{i+1}/K_i$ cyclic

$$Gal(L/F) = G_s \supseteq \cdots \supseteq G_0 = 1$$

Since $\mathrm{Gal}(K_{i+1}/K_i) = G_{i+1}/G_i$ cyclic, $\mathrm{Gal}(L/F)$ is solvable. Since K/F is Galois, $\mathrm{Gal}(K/F)$ is a quotient of $\mathrm{Gal}(L/F)$; hence solvable.

$\Leftarrow$: Let K be the splitting field for F.

$$\mathrm{Gal}(K/F) = G_s \supseteq \cdots \supseteq G_0 = 1 \qquad G_i/G_{i+1} \cong \mathbb{Z}/n_i\mathbb{Z}$$

$\Updownarrow$ Gal. corresp.

$$F = K_0 \subseteq \cdots \subseteq K_s = K \qquad K_{i+1}/K_i \text{ cyclic of deg } n_i$$

Let $F' = F(\zeta_{n_1}, \zeta_{n_2}, \ldots, \zeta_{n_s})$

Then,

$$\underbrace{F \subseteq F'}_{\text{abelian}} = F'K_0 \subseteq F'K_1 \underbrace{\subseteq}_{\text{cyclic of deg dividing } n_i} \cdots \subseteq F'K_s = F'K$$

Since $\mu_{n_i} \subseteq F'K_i$, all these ext'ns are simple radical, so $F'K$ is radical. $\square$

This proves the Abel-Ruffini Thm. (Cor. 40): the general deg. n poly., $n \geq 5$, is not solvable by radicals. But what about a specific poly.?

Ex: $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$ irred by Eis. ✓

K: splitting field, $G := \mathrm{Gal}(f)$

$5 \mid |G|$ since G transitive and $\{g \in G \mid g \cdot \alpha = \alpha\}$ is a subgp.

α root of f

This means that G contains a 5-cycle

$f(-2) = -17$ $f(0) = 3$ $f(1) = -2$ $f(2) = 23$ $\Rightarrow f$ has ≥ 3 real roots

$f'(x) = 5x^4 - 6$ has 2 real roots $\underset{MVT}{\Rightarrow}$ f has ≤ 3 real roots

By FTA, f has 5 roots $\rightarrow$ 2 nonreal roots

Let $\tau(z) = \overline{z}$. If α is a nonreal root,

$f(\tau(\alpha)) = f(\overline{\alpha}) = \overline{f}(\overline{\alpha}) = \overline{f(\alpha)} = \overline{0} = 0$, so

τ interchanges two roots of f and fixes the other 3; hence it is a transposition.

Any transposition & any 5-cycle generate S_5, so $G = S_5$ and f is not solvable by radicals.

Remark: cycle type important for computing Galois gp.

Sol'n of Cubic by Radicals (Cardano's Formula)

$$g(y) = y^3 + py + q$$
$$= (y-\alpha)(y-\beta)(y-\gamma)$$

$1 \triangleleft A_3 \triangleleft S_3$

A_3 cyclic

$S_3/A_3 \cong \mathbb{Z}_2$ cyclic

$s_1 = \alpha + \beta + \gamma = 0$

$D = -4p^3 - 27q^2$

Let $\rho := \rho_3 = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$

$F = \mathbb{Q}(\sqrt{D}, \rho) \leftarrow$ radical

Let

$$\theta_1 := (\alpha, \rho) = \alpha + \rho\beta + \rho^2\gamma$$

$$\theta_2 := (\alpha, \rho^2) = \alpha + \rho^2\beta + \rho\gamma$$

$$(\alpha, 1) = \alpha + \beta + \gamma = 0$$

So $\alpha = \frac{1}{3}(\theta_1 + \theta_2)$ $\quad \beta = \frac{1}{3}(\rho^2\theta_1 + \rho\theta_2)$ $\quad (*)$

$$\gamma = \frac{1}{3}(\rho\theta_1 + \rho^2\theta_2)$$

By Prop 37, $\theta_1^3, \theta_2^3 \in F$. Specifically,

$$\theta_1^3 = -\frac{27}{2}q + \frac{3}{2}\sqrt{-3D}$$

$$\theta_2^3 = -\frac{27}{2}q - \frac{3}{2}\sqrt{-3D} \qquad (**)$$

$(*)$ & $(**)$ give α, β, γ

<u>Note</u>: θ_1, θ_2 not indep; $\theta_1\theta_2 = -3p$,

so choice of $\theta_1 = \sqrt[3]{\theta_1^3}$ fixes choice of $\theta_2 = \sqrt[3]{\theta_2^3}$

Thm (Cardano & others, 1545):

$$x^3 + px + q = 0 \quad \text{has sol'ns}$$

$$x = \underbrace{\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}_{A} + \underbrace{\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}_{B} \qquad \text{s.t. } AB = -p$$

Sol'n of Quartic by Radicals

$$g(y) = y^4 + py^2 + qy + r = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

$$h(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2 = (x-\theta_1)(x-\theta_2)(x-\theta_3)$$

cubic resolvent

Cardano

$$\theta_1 = (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)$$

$$\theta_2 = (\alpha_1+\alpha_3)(\alpha_2+\alpha_4)$$

$$\theta_3 = (\alpha_1+\alpha_4)(\alpha_2+\alpha_3)$$

K: splitting field of g

E: splitting field of h

Claim: $\mathrm{Gal}(K/E) \cong V_4$

Since V_4 is solvable, and $E/\mathbb{Q}$ is radical by Cardano's formula, K is radical.

Pf of claim:

$$s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \text{ so } \alpha_1 + \alpha_2 = -(\alpha_3 + \alpha_4)$$

$$\theta_1 = -(\alpha_1 + \alpha_2)^2, \text{ so } \alpha_1 + \alpha_2 = \sqrt{-\theta_1}, \quad \alpha_3 + \alpha_4 = -\sqrt{-\theta_1}$$

Similarly,

$$\alpha_1 + \alpha_3 = \sqrt{-\theta_2}, \quad \alpha_2 + \alpha_4 = -\sqrt{-\theta_3}$$

$$\alpha_1 + \alpha_4 = \sqrt{-\theta_2}, \quad \alpha_2 + \alpha_3 = -\sqrt{-\theta_3}$$

Hence $\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3}$ symm.; turns out to $= -q$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$,

$$\sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3} = \alpha_1 + \alpha_2 + \alpha_1 + \alpha_3 + \alpha_1 + \alpha_4 = 2\alpha_1$$

Similar ideas give other roots:

$$\alpha_1 = \frac{1}{2}\left(\sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{\theta_3}\right)$$

$$\alpha_2 = \frac{1}{2}\left(\sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{\theta_3}\right)$$

$$\alpha_3 = \frac{1}{2}\left(-\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{\theta_3}\right)$$

$$\alpha_4 = \frac{1}{2}\left(-\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{\theta_3}\right)$$

Thus,

$$K = F\left(\sqrt{-\theta_1}, \sqrt{-\theta_2}\right)$$

$$\left(\text{since } \sqrt{-\theta_3} = \frac{-q}{\sqrt{-\theta_1}\sqrt{-\theta_2}}\right)$$

Hence

$$\mathrm{Gal}(K/F) \cong V_4$$

$\square$