H/w 6 due Tues. noon; H/w 7 will be posted soon

Final exam: Thurs. 3/23 8:30–11:30 Room 200-205 (email to come)

Degree 4

$$f(x) = x^4 + ax^3 + bx^2 + cx + d = g(y) := y^4 + py^2 + qy + r$$

$y = x + a/4 \qquad p = \frac{1}{8}(-3a^2 + 8b) \qquad q = \frac{1}{8}(a^3 - 4ab + 8c)$

$$r = \frac{1}{256}(-3a^4 + 16a^2b - 64ac + 256d)$$

roots: $\alpha, \beta, \gamma, \delta \qquad G := \mathrm{Gal}(g)$, K = splitting field of g

If $g(y)$ = linear · cubic, see cubic case above

If $g(y)$ = irred. quad · irred quad., $K = F(\sqrt{D_1}, \sqrt{D_2})$

If $\frac{\sqrt{D_1}}{\sqrt{D_2}} \in F$, $K = F(\sqrt{D_1})$, $G = \mathbb{Z}/2\mathbb{Z}$

otherwise, $G = K_4$

Now assume g irred.

Since G transitive, $G \leq S_4$, must have

G = one of: S_4, A_4,

$D_8 = \langle (1324), (13)(24) \rangle$, or $\sigma D_8 \sigma^{-1}$,

$V_4 = \langle (12)(34), (14)(23) \rangle$, or $C = \langle (1324) \rangle$, or $\sigma C \sigma^{-1}$

Important tool: <u>resolvent</u> cubic

Let
$$\left.\begin{array}{ll} \Theta_1 = (\alpha+\beta)(\gamma+\delta) & \leftarrow \text{Fixed by } D_8 \\ \Theta_2 = (\alpha+\gamma)(\beta+\delta) & \leftarrow \text{Fixed by another } D_8 \\ \Theta_3 = (\alpha+\delta)(\beta+\gamma) & \leftarrow \text{Fixed by another } D_8 \end{array}\right\} \text{Fixed by } K_4$$

$s_1(\Theta_1, \Theta_2, \Theta_3) = 2p$ $\quad s_2(\Theta_1, \Theta_2, \Theta_3) = p^2 - 4r$

$s_3(\Theta_1, \Theta_2, \Theta_3) = -q^2$

So $h(x) := (x - \Theta_1)(x - \Theta_2)(x - \Theta_3) = x^3 - 2px^2 + (p^2 - 4r)x + q^2$

$$\left.\begin{array}{l} \Theta_1 - \Theta_2 = -(\alpha-\delta)(\beta-\gamma) \\ \Theta_1 - \Theta_3 = -(\alpha-\gamma)(\beta-\delta) \\ \Theta_2 - \Theta_3 = -(\alpha-\beta)(\gamma-\delta) \end{array}\right\} \text{prod}^2 \text{ of these } = D \quad (!)$$

So disc of g = disc of $h = 16p^4r - 4p^3q^2$

$-128p^2r^2 + 144pq^2r - 27q^4 + 256r^3$

Splitting field of $h \subseteq$ Splitting field of g

Cases:

A) h irred, $\sqrt{D} \notin F$.

$G \nsubseteq A_4$

$\text{Gal}(h) \nsubseteq A_3$ (so $\text{Gal}(h) = S_3$)

So $G = S_4$

B) h irred, $\sqrt{D} \in F$.

$G \subseteq A_4$

$\text{Gal}(h) = A_3$

$|G| \geq \text{lcm}(4,3) = 12 = |A_4|$

so $G = A_4$

C) $h = \text{linear} \cdot \text{linear} \cdot \text{linear}$

$\theta_1, \theta_2, \theta_3 \in F = \text{Fix}\, G$

So $G \subseteq K_4$ and since $|G| \geq 4$, $G = K_4$

D) $h = \text{linear} \cdot \text{irred. quad}$

One of $\theta_1, \theta_2, \theta_3 \in F$, say θ_1

$G \nsubseteq K_4$

$G \subseteq D_8$

$|G| \geq 4$

So $G = D_8$ or $G = C$

Claim: $G = D_8$ iff $g(y)$ irred over $F(\sqrt{D})$

Pf: $F(\sqrt{D}) = \mathrm{Fix}(G \cap A_4)$

$D_8 \cap A_4 = K_4$ transitive on roots $\to g$ irred.

$C \cap A_4 = \mathbb{Z}/2\mathbb{Z}$ not trans on roots $\to g$ red.

Fundamental Thm. of Algebra (Thm 35): $\mathbb{C}$ is alg. closed.

Pf (Artin):

Claim 1: Every poly. over $\mathbb{R}$ w/ odd degree has a root in $\mathbb{R}$

Pf: Intermediate value thm

Claim 2: Every quadratic poly over $\mathbb{C}$ has a root in $\mathbb{C}$

Pf: Quadratic formula.

Let $f(x) \in \mathbb{R}$ be a poly. of deg n, w/ splitting field K. Then, $K(i)$ is the splitting fiels of $f(x) \cdot (x^2+1)$, so $K(i)/\mathbb{R}$ is Galois. Let H be a Sylow 2-subgp. of $G := \mathrm{Gal}(K(i)/\mathbb{R})$. $[\mathrm{Fix}\, H : \mathbb{R}] = |G : H|$ is odd, so by Claim 1 equals 1.

Thus, $G = H$ is a 2-group, so $G' := Gal(K(i)/\mathbb{C})$ is also a 2-group. All nontrivial 2-groups have a subgp. of index 2, so let H' be such a group. Then Fix H' is a deg 2 ext'n of $\mathbb{C}$, which contradicts Claim 2. Therefore, $K(i) = \mathbb{C}$, so every poly. over $\mathbb{R}$ has a root in $\mathbb{C}$. Thus, $\mathbb{C}$ is the alg. closure of $\mathbb{R}$, so it is alg. closed. $\square$

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§14.7: Insolvability of the Quintic

Recall: A finite group G is solvable if $\exists$

$$1 = G_s \lhd G_{s-1} \lhd \cdots \lhd G_0 = G$$

s.t. G_i / G_{i+1} is cyclic

Def: $f(x) \in F[x]$ can be solved by radicals if $\exists$

$$F_0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_s = K$$

s.t. $K_{i+1} = K(\sqrt[n_i]{a_i})$ for some $a_i \in K_i$

Thm 39: Let $f(x) \in F[x]$ w/ char $F = 0$. Then $f(x)$ can be solved by radicals $\iff$ $Gal(f)$ is solvable.

Cor 40: The general poly. of deg $n \geq 5$ is not solvable by radicals

Pf: S_n, $n \geq 5$ is not solvable since A_n is simple (and not cyclic.