Math 121, Winter 2023, Homework 1 Solutions

Section 13.1

Problem 1. *Show that* $p(x) = x^3 + 9x + 6$ *is irreducible in* $\mathbb{Q}[x]$*. Let* θ *be a root of* $p(x)$*. Find the inverse of* $1 + \theta$ *in* $\mathbb{Q}(\theta)$*.*

Solution. By Proposition 10 in Chapter 9 of Dummit and Foote, $p(x)$ is reducible in $\mathbb{Q}[x]$ if and only if it has a root in $\mathbb{Q}$. By the Rational Root Theorem (D&F Ch.9, Proposition 11), if $r = a/b \in \mathbb{Q}$, then a divides the constant term, 6, of p, and b divides the coefficient, 1, of the top degree term of p. Therefore, r must equal $\pm 1, \pm 2, \pm 3$, or ± 6. Plugging these values into $p(x)$ shows that none of them are roots, so p is irreducible.

Alternatively, we can use Eisenstein's criterion. All coefficients in $p(x)$ are divisible by 3 except for the top degree term, and the constant term is not divisible by 9. Therefore, $p(x)$ satisfies the hypotheses of Eisenstein's criterion, so is irreducible.

Now, the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$ is some $\mathbb{Q}$-linear combination $a + b\theta + c\theta^2$ of $1, \theta$, and θ^2, since these form a basis for $\mathbb{Q}(\theta)$ over $\mathbb{Q}$. Since θ is a root of p, $\theta^3 = -9\theta - 6$, so

$$(1+\theta)(a + b\theta + c\theta^2) = a + (a+b)\theta + (b+c)\theta^2 + c\theta^3 = a - 6c + (a + b - 9c)\theta + (b+c)\theta^2.$$

For $a + b\theta + c\theta^2$ to be the inverse of $1 + \theta$, this expression must equal 1, and solving the resulting system of equations gives $a = \frac{5}{2}, b = -\frac{1}{4}, c = \frac{1}{4}$, so $\theta^{-1} = \frac{5}{2} - \frac{1}{4}\theta + \frac{1}{4}\theta^2$.

Problem 2. *Show that* $p(x) = x^3 - 2x - 2$ *is irreducible over* $\mathbb{Q}$ *and let* θ *be a root. Compute* $(1+\theta)(1+\theta+\theta^2)$ *and* $\frac{1+\theta}{1+\theta+\theta^2}$ *in* $\mathbb{Q}(\theta)$*.*

Solution. Again, use the rational root theorem to show that $p(x)$ doesn't have a root in $\mathbb{Q}$, and is therefore irreducible. Alternatively, use Eisenstein's criterion with the prime 2.

Since θ is a root of p, $\theta^3 = 2\theta + 2$, so

$$(1+\theta)(1+\theta+\theta^2) = 1 + 2\theta + 2\theta^2 + \theta^3 = 3 + 4\theta + 2\theta^2.$$

For the final part, let $a + b\theta + c\theta^2 = \frac{1+\theta}{1+\theta+\theta^2}$. Then,

$$\begin{aligned}
1 + \theta &= (a + b\theta + c\theta^2)(1 + \theta + \theta^2) \\
&= a + (a + b)\theta + (a + b + c)\theta^2 + (b + c)\theta^3 + c\theta^4 \\
&= a + (a + b)\theta + (a + b + c)\theta^2 + (b + c)(2\theta + 2) + c(2\theta^2 + 2\theta) \\
&= a + 2b + 2c + (a + 3b + 4c)\theta + (a + b + 3c)\theta^2.
\end{aligned}$$

Solving this system of equations gives

$$\frac{1 + \theta}{1 + \theta + \theta^2} = \frac{1}{3}(1 + 2\theta - \theta^2).$$

Problem 3. *Show that $p(x) = x^3 + x + 1$ is irreducible over $\mathbb{F}_2$ and let θ be a root. Compute the powers of θ in $\mathbb{F}_2(\theta)$.*

Solution. Once again, $p(x)$ is irreducible unless it has a root. $\mathbb{F}_2$ only has two elements, so we plug them both in and find that neither is a root: $p(0) = p(1) = 1$. Therefore, $p(x)$ is irreducible.

To compute the powers of θ, note that θ is a root of p, so $\theta^3 = -\theta - 1 = \theta + 1$, since in $\mathbb{F}_2$, 1 and -1 are equal!. Then

$$\theta^4 = \theta(\theta + 1) = \theta^2 + \theta, \qquad \theta^5 = \theta(\theta^2 + \theta) = \theta^2 + \theta + 1,$$

$$\theta^6 = \theta(\theta^2 + \theta + 1) = \theta^2 + 1, \qquad \theta^7 = \theta(\theta^2 + 1) = 1,$$

and the powers of θ repeat from there via the relationship $\theta^n = \theta^7\theta^{n-7} = \theta^{n-7}$, so that $\theta^{7i+j} = \theta^j$.

Problem 4. *Prove directly that the map $a + b\sqrt{2} \mapsto a - \sqrt{2}$ is an isomorphism of $\mathbb{Q}(\sqrt{2})$ with itself.*

Solution. Let φ be the given map. Since $\varphi(\varphi(a + b\sqrt{2})) = \varphi(a - b\sqrt{2}) = a + b\sqrt{2}$, $\varphi \circ \varphi$ is the identity map, so it is its own inverse, and therefore a bijection.

The following computations show that φ is a ring homomorphism:

$$\varphi((a+b\sqrt{2})+(c+d\sqrt{2})) = \varphi(a+c+(b+d)\sqrt{2}) = a+c-(b+d)\sqrt{2} = \varphi(a+b\sqrt{2})+\varphi(c+d\sqrt{2}),$$

$$\varphi((a+b\sqrt{2})(c+d\sqrt{2})) = \varphi(ac+2bd+(ad+bc)\sqrt{2}) = ac+2bd-(ad+bc)\sqrt{2} = \varphi(a+b\sqrt{2})\varphi(c+d\sqrt{2}),$$

and we don't need to check that $\varphi(1) = 1$ since $\mathbb{Q}(\sqrt{2})$ is a field.

Note that φ is the isomorphism given in Theorem 8.

Problem 6. *Show that if* α *is a root of* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *then* $a_n\alpha$ *is a root of the monic polynomial* $q(x) = x^n + a_{n-1}x^{n-1} + a_n a_{n-2}x^{n-2} + \cdots + a_n^{n-2}a_1 x + a_n^{n-1}a_0$.

Solution. This follows from the fact that $q(a_n x) = a_n^{n-1}p(x)$.

Problem 7. *Prove that* $p(x) = x^3 - nx + 2$ *is irreducible in* $\mathbb{Q}[x]$ *for* $n \neq -1, 3, 5$.

Solution. Once again, since $p(x)$ is a cubic, it is reducible precisely when it has a root, and by the rational root theorem that root must be one of: $-2, -1, 1, 2$. Plugging these values in:

$$p(-2) = -6 + 2n, \qquad p(-1) = 1 + n, \qquad p(1) = 3 - n, \qquad p(2) = 10 - 2n.$$

All of these quantities are nonzero unless $n = -1, 3$, or 5.