We show equivalence of four definitions for constructability of complex numbers.

Let $L \subseteq \mathbb{C}$ be the smallest field containing $\mathbb{Q}$ such that $\beta \in L \Rightarrow \sqrt{\beta} \in L$.

Prop: Let $\alpha = x + iy = re^{i\theta}$

The following are equivalent:

a) x, y are constructible

b) r, $\cos\theta$ are constructible

c) $\exists K$, $\alpha \in K$, s.t

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K, \quad [K_i : K_{i-1}] = 2$$

d) $\alpha \in L$

We assume these are equiv. when $\alpha \in \mathbb{R}$ (§13.3)

Pf: a) $\Rightarrow$ b): $r = \sqrt{x^2+y^2}$, $\cos\theta = \dfrac{x}{r}$

b) $\Rightarrow$ a): $x = r\cos\theta$, $y = r\sin\theta$

c) $\Rightarrow$ d): Each deg. 2 ext'n is of the form $K_i = K_i(\sqrt{D_i})$, so $K \subseteq L$.

d) $\Rightarrow$ c): Let L' be the set of all $\beta \in \mathbb{C}$ w/ property c). We show that $L \subseteq L'$ be showing that L' is a field closed under square roots. If $\beta, \gamma \in L'$, suppose $\beta \in K$, $\gamma \in E$ with

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K, \quad [K_i : K_{i-1}] = 2$$

$$\mathbb{Q} = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E, \quad [E_i : E_{i-1}] = 2$$

Then, $\beta, \gamma \in KE$, and

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K = KE_0 \subseteq KE_1 \subseteq \cdots \subseteq KE_n = KE$$

We have $[K_i : K_{i-1}] = 2$ and $[KE_i : KE_{i-1}] \leq [E_i : E_{i-1}] = 2$. Removing the redundant fields gives the desired sequence of fields for KE, so $KE \subseteq L'$ and $\beta + \gamma$, etc. $\in L'$, so L' is a field.

Also, $\sqrt{\beta} \in K(\sqrt{\beta})$, which has degree 1 or 2 over K, so $\sqrt{\beta} \in L'$. Therefore, $L \subseteq L'$, so $\alpha \in L'$. (In fact, $L = L'$).

a) $\Rightarrow$ d): $\operatorname{Re} \alpha$, $\operatorname{Im} \alpha \in L$ by assumption, and $i \in L$ since $-1 \in \mathbb{Q} \subseteq L$. Therefore $\alpha = \operatorname{Re} \alpha + i \operatorname{Im} \alpha \in L$.

d) $\Rightarrow$ a): If $\alpha \in L$, $\overline{\alpha} \in L$ since the definition of L is invariant under the autom. $i \mapsto -i$. In other words, if $L^c = \{\bar{z} \mid z \in L\}$, then L^c is a field closed under taking square roots, so $L \cap L^c$ is one as well. Since L is the smallest such field, we must have $L = L \cap L^c$ i.e. $L = L^c$. Therefore,

$$\mathrm{Re}\,\alpha = \tfrac{1}{2}(\alpha + \bar{\alpha}), \quad \mathrm{Im}\,\alpha = \tfrac{1}{2}(\alpha - \bar{\alpha}) \in L \qquad \square$$